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Tsallis–Cirto entropy of black hole and black hole atom

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The quantum tunneling processes related to the black hole determine the black hole thermodynamics. The Hawking temperature is determined by the quantum tunneling processes of radiation of particles from the black hole. On the other hand, the Bekenstein-Hawking entropy of the black hole is obtained by consideration of the macroscopic quantum tunneling processes of splitting of black hole to the smaller black holes. These tunneling processes also determine the composition rule for the black hole entropy, which coincides with the composition rule for the non-extensive Tsallis–Cirto $\delta = 2$ entropy. This composition rule suggests that the mass spectrum of the black hole is equidistant, $M = NM_0$. Here N is an integer number and $M_0 = \sqrt{2}m_{\rm P}$ is the mass quantum expressed via the reduced Planck mass $m_{\rm P}$. The Bekenstein–Hawking entropy of the black hole with mass $M = NM_0$ is $S_{\rm BH}(N) = N^2$.

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The Bekenstein–Hawking area law for the black hole entropy is non-extensive, and there are many attempts to find possible alternatives to the Bekenstein–Hawking entropy using the Renyi and Tsallis statistics and the other types of generalized entropy, see [1–4] and references therein. In particular, the non-additivity of the Bekenstein–Hawking entropy of the Schwarszchild black holes, $S_{\rm BH}(M) = 4\pi GM^2$, was treated in terms of the non-extensive Tsallis–Cirto entropy [5–7]. Here we consider the black hole configurational space, where only processes between the black holes are taken into account, i.e. without any matter components. We show that the rare quantum tunneling processes of the splitting of the black hole into smaller parts provide the non-extensive composition law.

The semiclassical tunneling description of the Hawking radiation of particles from the black holes [8–10] provides the Hawking temperature, which determines is the rate of emission. Parikh and Wilczek [8] obtained the correction to the Hawking radiation caused by the back reaction – the reduction of the black hole mass after emission. This back reaction is related to the decrease of the Bekenstein–Hawking entropy $S_{\rm BH}(M) = 4\pi GM^2$ after emission [11]. One can consider the Hawking radiation as the rare effect caused by thermodynamic fluctuations, while the latter can be described in terms of entropy change. Then the rate of Hawking radiation can be described in terms of the difference of the black hole entropies before and after emission of a particle. This can be extended to consideration of the macroscopic quantum tunneling which describes the splitting of the black hole [12].

As distinct from the radiation of the point particle, the rate of emission of a small black hole is enhanced by the entropy of the emitted black hole [13]. This shows that the processes which involve the black holes in the absence of matter, such as splitting and merging of the black holes, are determined by the entropies of black holes. These processes form the configurational space, which is the source of the non-extensive entropy.

This demonstrates the reason, why the black hole entropy does not satisfy the additivity condition, $S(A,B) = S(A) + S(B)$, which is valid for the thermodynamics of the extensive systems. In the systems, which obey the extensive thermodynamics, the entropy is proportional to the volume of the system. The splitting of the system with volume V in two parts with volumes $V_1 + V_2 = V$ does not change the total entropy of the system, $S(V_1 + V_2) = S(V_1) + S(V_2)$.

On the contrary, the splitting of the black hole into smaller black holes is the consequence of the rare processes of the macroscopic quantum tunneling transition, which rate is determined by the entropy change. Calculating these rates, one comes to the non-additive composition rule for the black hole entropies:

$$S_{\rm BH}(M_1 + M_2) = \left(\sqrt{S_{\rm BH}(M_1)} + \sqrt{S_{\rm BH}(M_2)}\right)^2. \quad (1)$$

The macroscopic quantum tunneling approach is actually another way of the derivation of the Bekenstein–Hawking entropy of the black hole.

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Such processes require the generalization of the statistics with the corresponding non-extensive entropy. The Equation (1) fully determines the type of the statistics. It corresponds to the special class of the non-extensive entropies – the Tsallis–Cirto δ-entropy [5–7]:

$$S_\delta = \sum_{i=1}^{n} p_i \left(\ln \frac{1}{p_i}\right)^\delta , \qquad (2)$$

where $\sum_{i=1}^{n} p_i = 1$. For the Tsallis–Cirto entropy S_δ, there is the following composition rule for two independent systems A and B with $n_{A+B} = n_A n_B$:

$$S^{1/\delta}_{\delta,A+B} = S^{1/\delta}_{\delta,A} + S^{1/\delta}_{\delta,B} . \qquad (3)$$

For $\delta = 2$, this corresponds to the composition rule for entropies in Eq. (1). This statistics is applicable to the entropy of black hole as the finite closed system and cannot be applied to such open systems as the de Sitter state, where the entropy is extensive [14, 15].

Let us assume that all the black-hole micro-states are equally probable with the probabilities

$$p_i = \frac{1}{n} = \exp\left(-\frac{M}{\sqrt{2} m_{\rm P}}\right) = \exp\left(-\frac{M}{M_0}\right) \equiv e^{-N} , \qquad (4)$$

where $m_{\rm P} = 1/\sqrt{8\pi G}$ is the reduced Planck mass. Then Equation (2) with $\delta = 2$ gives the black hole entropy $4\pi G M^2$:

$$S_{\delta=2} = \left(\ln \frac{1}{p_i}\right)^2 = 4\pi G M^2 = S_{\rm BH}(M) = N^2 . \qquad (5)$$

Equation (4) can be interpreted in terms of the black-hole quantum – the black hole with the mass $M_0 = \sqrt{2} m_{\rm P}$, i.e. the black hole mass is quantized, $M = N M_0$. This is valid for the black hole with arbitrary mass M, since the mass quantum M_0 does not depend on M. This suggests that the black hole thermodynamics can be described in terms of the ensemble of micro black holes. The black holes with the Planck scale masses have been discussed by Hawking [16], see also [17–22] and Markov's maximons [23].

The quantization of the black hole mass, $M = N M_0$, differs from the Bekenstein quantization of entropy [12, 24–33]. In the Bekenstein approach, the entropy is the adiabatic invariant, which spectrum is equally spaced, $S_{\rm BH} = aN$. Here a is the dimensionless parameter, which value depends on the microscopic theory. The corresponding mass spectrum in this approach is $M \propto m_{\rm P}\sqrt{N}$. The quantization inspired by the string theory [34, 35] gives the square-root rule, $S = 2\pi(\sqrt{N_1} \pm \sqrt{N_2})$.

In our approach, where only the black hole configurations are considered, the black hole entropy is quadratic, $S_{\rm BH} = N^2$, while the mass spectrum is equidistant. Although the description of the black hole in terms of elementary black holes looks rather artificial, it properly describes the black hole thermodynamics. On the other hand, if such description does really reflect the physical structure of the black hole interior, the quantization of the black hole mass will highly suppress the Hawking radiation of the ordinary matter.

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