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Comment on "Topological origin of horizon temperature via the Chern–Gauss–Bonnet theorem" (Pis'ma v ZhETF 122, 199 (2025))

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There is a difference between the temperature attributed to the Hawking radiation from the cosmological horizon in the de Sitter (dS) state [1] (the Gibbons–Hawking temperature $T_{\rm GH} = H/2\pi$) and the twice as high local temperature of the dS state $T = H/\pi = 2T_{\rm GH}$, see [2, 3] and references therein. Recently it was suggested that this difference has also a geometrical origin. In this topological approach, the temperature is determined by the Euler characteristic $\chi(\mathcal{M})$ of the spacetime with the Euclidean time [4]. The topological Euler characteristic, which is determined by Chern–Gauss–Bonnet (CGB) invariant, is different for different manifolds chosen for the consideration of the dS thermodynamics. If the global dS manifold S^4 is chosen, then its Euler characteristic is $\chi(S^4) = 1$. If the manifold is considered, which is restricted to the near-horizon region outside the horizon, its topological invariant is $\chi(D^2 \times S^2) = 2$. This gives the geometric support to the factor 2 difference between the two temperatures. The temperature of the Hawking radiation from the cosmological horizon is determined by the near-horizon topology. This leads to the Hawking temperature, $\beta = \pi\chi(D^2 \times S^2)/H = 2\pi/H$. On the other hand, local processes, such as the ionization of an atom in the dS environment, occur in bulk and have no relation to the horizon. They are described by the global S^4 topology, and the corresponding temperature of the dS state is twice larger, $\beta = \pi\chi(S^4)/H = \pi/H$.

This geometric approach was also applied to the black holes [5–7], where the topology of the near-horizon region outside the black hole determines the temperature of the Hawking radiation. However, the application to the Reissner–Nordström (RN) black hole with two horizons in [5, 7] is in contradiction with our result for the thermodynamics of RN black hole [8–10], that the entropy of the RN black hole depends only on its mass. The source of the contradiction is that only the topology associated with the outer horizon was considered, while the topology of the inner horizon was ignored. Here we include into consideration both horizons and obtain the agreement with [8–10].

We use expression for the CGB invariant $\chi(\mathcal{M})$ in [7], but apply to the outer horizon at $r = r_+$ and to the inner horizon at $r = r_-$, which have different inverse temperatures, β_+ and β_- determined by the corresponding CGB invariant. The invariant for the outer horizon [7]:

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2}\int_0^{\beta_+} d\tau \int_{r_+}^{\infty} dr \int_{\Sigma} \sqrt{g} d^2x \mathcal{R}, \tag{1}$$

where $\mathcal{R} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu}$. It provides the connection between the radius r_+ and the corresponding temperature β_+. The topological invariant for the inner horizon provides the connection between its radius r_- and the corresponding temperature β_-:

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2}\int_0^{\beta_-} d\tau \int_{r_-}^{\infty} dr \int_{\Sigma} \sqrt{g} d^2x \mathcal{R}\,. \tag{2}$$

For the RN black hole one has:

$$\mathcal{R} = \frac{8}{r^8}(6M^2r^2 - 12MQ^2r + 5Q^4), \tag{3}$$

where $r_+ + r_- = 2M$ and $r_+r_- = Q^2$ ($G = \hbar = 1$). With $\chi(\mathcal{M}) = 2$, Eqs. (1) and (2) give:

$$\beta_+ = 4\pi\frac{r_+^2}{r_+ - r_-} = \frac{2\pi r_+^2}{\sqrt{M^2 - Q^2}}, \tag{4}$$

$$\beta_- = 4\pi\frac{r_-^2}{r_- - r_+} = -\frac{2\pi r_-^2}{\sqrt{M^2 - Q^2}}\,. \tag{5}$$

The negative temperature for inner horizon in Eq. (5) is in agreement with [11]. Thus, the topological consideration is applicable to both horizons.

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The thermodynamics of RN black hole is determined by both horizons. Example is provided by the Hawking radiation, which represents the coherent sequence of processes of tunnelling across the two horizons, each determined by gravity at the corresponding horizon. The total radiation rate of this coherent process is

$$w \propto e^{-\beta_+ E} e^{-\beta_- E} = e^{-(\beta_+ + \beta_-)E} = e^{-\beta E}. \qquad (6)$$

The temperature $\beta = 4\pi(r_+ + r_-) = 8\pi M$ [8–11], which describes this co-tunneling process, does not depend on charge and differs the Hawking temperature related to the outer horizon. The entropy of the RN black hole also does not depend on the charge [8, 9]: $S_{\rm RN} = 4\pi M^2$. This is supported by the adiabatic process: the RN black hole can be adiabatically transformed to the Schwarzschild black hole by varying the fine structure constant α at fixed mass M. The entropy does not change, and thus it is the same for the RN and Schwarzschild black holes [9]. This can be also obtained from the non-extensive Tsallis–Cirto statistics [12], which is applicable to the processes of merging and splitting of black holes [13, 14]. It gives the composition rule for entropy of the RN black hole in terms of entropies of two horizons, $S(r_\pm) = \pm\pi r_\pm^2$ [15]:

$$S = \left(\sqrt{S(r_+)} + \sqrt{|S(r_-)|}\right)^2 = \pi(r_+ + r_-)^2 = 4\pi M^2. \qquad (7)$$

Equation (7) is valid up to an extreme point, where $r_- = r_+$ and two horizons merge and annihilate. At the transition point, continuity is broken [16], violating adiabatic and topological approaches. In [6, 7] it was suggested that for the extremal RN black hole the CGB invariant is zero. If so, then at the transition point the topological invariant changes abruptly, and the entropy jumps from $S = 4\pi M^2$ to $S = 0$. Such jumps take place at quantum phase transitions between states with different topological invariants [17, 18].

This approach can be also applied to the horizon of the white hole. Using the anti-symmetry between the black and white holes, we can consider the white hole as a mirror of the black hole at negative r (or a quantum clone in 't Hooft formulation [19]). Then the topological invariant for the outer horizon of the white hole gives the inverse temperature $\beta_+(\text{white}) = -\beta_+(\text{black}) < 0$. Correspondingly for the inner horizon of a white hole we have $\beta_-(\text{white}) = -\beta_-(\text{black}) > 0$. As a result, the entropy of the white hole is with minus sign the entropy of the black hole with the same mass. Such anti-symmetry of entropies is supported by the process of macroscopic quantum tunnelling from black hole to white hole. The calculations of the tunnelling exponent gives the following rate of this process: $w \sim \exp(-2S_{\rm BH})$, where $S_{\rm BH}$ is the entropy of black hole [9]. The same result was obtained for the tunnelling of the black hole to white hole in Jackiw–Teitelboim gravity [20]. On the other hand, this quantum tunneling represents the rare quantum fluctuation, and thus its rate is determined by the difference between the entropies of two macroscopic objects, $w \sim \exp(\Delta S)$, where $\Delta S = S_{\rm WH} - S_{\rm BH}$. This gives $S_{\rm WH} = -S_{\rm BH}$.

It is shown that the topological approach to black hole thermodynamics and dS spacetime proposed in [4] also works for RN black holes with two horizons. Both horizons with their near-horizon topology contribute to the thermodynamics of the RN black hole. The correlation between the horizons makes the entropy of the RN black hole independent of its electric charge. The confirmation of the geometric approach is important for dS thermodynamics. It supports the topological explanation of the difference between the Gibbons–Hawking temperature $T_{\rm GB} = H/2\pi$, which is determined by the near-horizon topology, and the local dS temperature $T = H/\pi$, which is determined by the global dS topology [4].

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