Pis'ma v ZhETF, vol. 122, iss. 7, pp. 432 – 433

Reply to the Comment on "Topological origin of horizon temperature via the Chern–Gauss–Bonnet theorem" (Pis'ma v ZhETF 122, 199 (2025))

J. C. M. Hughes[1)], *F. V. Kusmartsev*[1)]

College of Engineering and Physical Sciences, Khalifa University, PO Box 127788, Abu Dhabi, United Arab Emirates

Submitted 26 August 2025
Resubmitted 26 August 2025
Accepted 29 August 2025

DOI: 10.31857/S0370274X25100108, EDN: ZJYAEK

Recently, Jack C. M. Hughes and Fedor V. Kusmartsev reported on a topological approach to the horizon temperature of de Sitter and Schwarzschild spacetime [1, 2]. This technique involves the analytic continuation $t \to i\tau$ of the geometry, followed by the necessary compactification in τ to remove singularities. One then computes the topological Euler characteristic of the manifold, using the Chern–Gauss–Bonnet theorem [3, 4]

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2} \times \int_{\mathcal{M}} \sqrt{-g} d^4x \left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right), \tag{1}$$

leading to a topological source of the Hawking temperature

$$T_H \propto \frac{1}{\chi(\mathcal{M})} \tag{2}$$

in both geometries. Grigory E. Volovik has followed this with a successful extension of the approach to Reissner–Nordström spacetime [5]. This is particularly important since it verifies the applicability of the topological argument to *multi-horizon spacetimes*. Furthermore, the presentation of [5] clarifies the validity of (1)–(2) in the case of black hole spacetimes and raises relevant questions on the thermodynamics of Reissner–Nordström. We comment here on these points.

Strictly speaking, the application of (1) requires the manifold $\mathcal{M}$ under consideration to be compact (with or without boundary) [4]. In the case of de Sitter, this condition is satisfied since one obtains the 4-sphere upon analytic continuation. If one instead works in the static patch

$$ds^2 = -\left(1 - \frac{r^2}{l^2}\right)d\tau^2 + \left(1 - \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{3}$$

then the Euclidean geometry of interest has topology $D^2 \times S^2$. In the actual application of (1) however, one is interested in integrating only over the compact domain $[0, l]$ in r (from the horizon *inwards*), retaining consistency (and recovering the Gibbons–Hawking temperature $T_{\rm GH} = 1/\chi(D^2 \times S^2)\pi l)$ [1, 2]. For Schwarzschild and Reissner–Nordström however the situation is more delicate. Their analytic continuation again has topology $D^2 \times S^2$ (after τ compactification), but now one seeks to apply (1) over the non-compact domain $[r_h, \infty)$ (from the horizon *outwards*). Nevertheless, one still obtains the appropriate Gibbons–Hawking temperature in following this procedure [1, 2, 5]. The reason this works is the following: the theorem (1) is restricted to the compact case to ensure convergence of $\chi(\mathcal{M})$ to an integer [4]. This does not mean that the non-compact case must diverge, rather that its application is dependent upon the geometry under consideration. For both Schwarzschild and Reissner–Nordström the integrand in (1) falls-off sufficiently fast to guarantee convergence:

$$\sqrt{-g}\, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim \frac{1}{r^4} \tag{4}$$

for Schwarzschild, and

$$\sqrt{-g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu}\right) \sim \frac{1}{r^6} - \frac{1}{r^5} + \frac{1}{r^4} \tag{5}$$

for Reissner–Nordström. Hence, while the geometries are non-compact (1) can be consistently applied to yield (2) due to the functional dependence of the characteristic class [6, 7] (1) in r.

Focusing now on Reissner–Nordström

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)d\tau^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{6}$$

Volovik has demonstrated ([5] and references therein) that both the outer r_+ and inner r_- horizons are of

[1)]e-mail: jack.hughes.phys.14@gmail.com; fedor.kusmartsev@ku.ac.ae

relevance when determining the thermodynamics of the spacetime. From the topological perspective, the thermodynamic temperature of Reissner–Nordström is the consequence of an additivity,

$$\beta_{\rm RN} = \beta_+ + \beta_- = 8\pi M, \tag{7}$$

where

$$\chi(D^2 \times S^2) = \frac{\beta_\pm}{32\pi^2} \times \tag{8}$$
$$\times \int_{r_\pm}^{\infty} dr \int_\Sigma \sqrt{-g}\, d^2x \big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu}\big)$$

defines the inverse temperature of each respective horizon. Such an additivity property is characteristic of topological invariants ($\chi(\mathcal{M})$ under gluing) and adds support to horizon thermodynamics being topological. In other words, the total temperature β_{RN} requires an overlapping of (1) over different coordinate charts (defined by $r_\pm$), with each neighborhood contributing additively to global invariants and hence the thermodynamic temperature. In the limit of 0 overlap ($r_+ = r_-$) this picture collapses and the horizons annihilate signaling a topological phase transition [5]. While this was argued from the point of view of $\chi(\mathcal{M}) = 0$ for the extreemal limit, one can instead note that analytic continuation of the metric (6) when $r_+ = r_-$ is topologically a product space of Euclidean–Anti–de Sitter and the 2-sphere. There is no singularity to be removed in this space and hence no need to compactify τ, removing any notion of temperature: this is the source of the entropy collapse.

It is interesting to note that the compact interior region bound by $[r_-, r_+]$ does not admit a thermodynamic description, hence why it does not contribute in (2). Topologically the reason for this is clear: after analytic continuation $t \to i\tau$ this region becomes the product of the *annulus* and the 2-sphere. The annulus has vanishing Euler characteristic χ, preventing (1) from being well defined without the addition of a boundary term (which will be a Chern–Simons form) [4, 6, 7]. The construction and analysis of such a boundary term is important, as its behavior under the extremal limit $r_- = r_+$ will define entropy collapse, since in this limit the annulus collapses to a circle and the boundary is eliminated. This will be the subject of future research.

Funding. This work received institutional funding from Khalifa University. Jack C. M. Hughes and Fedor V. Kusmartsev acknowledge support from the Khalifa University Award Numbers FSU-2021-030/8474000371 as well as Research and Innovation Grants RIG-2023-028, RIG-2024-26, and RIG-2024-053. Fedor V. Kusmartsev also acknowledges support from the Thousand Talents Program and the President's International Fellowship Initiative of the Chinese Academy of Sciences Awards.

Conflict of interest. The authors of this work declare that they have no conflicts of interest.

1. J. C. M. Hughes and F. V. Kusmartsev, JETP Lett. **122**(4), 208 (2025); URL http://jetpletters.ru/ps/dx/10.31857/S0370274X25080136.
2. J. C. M. Hughes and F. V. Kusmartsev, Pis'ma v ZhETF **122**(4), 199 (2025); URL https://link.springer.com/article/10.1134/S0021364025607602.
3. S. S. Chern, Ann. Math. **45**, 747 (1944).
4. S. S. Chern, Ann. Math. **46**, 674 (1945).
5. G. E. Volovik, arXiv:2505.20194 (2025); Preprint 2505.20194.
6. L. W. Tu, *Differential Geometry: Connections, Curvature, and Characteristic Classes (Graduate Texts in Mathematics*, v. 275), Switzerland, Springer, Cham (2017); ISBN 978-3-319-55082-4.
7. E. W. Mielke, *Geometrodynamics of Gauge Fields: On the Geometry of Yang-Mills and Gravitational Gauge Theories*, Mathematical Physics Studies, 2nd ed., Springer, Cham (2017); ISBN 978-3-319-29733-0.