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Topological invariant responsible for the stability of the Fermi surfaces in non-homogeneous systems[1)]

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The topological invariant responsible for the stability of Fermi point/Fermi surface in homogeneous systems is expressed through the one particle Green function, which depends on momentum. It is given by an integral over the 3D hypersurface in momentum space surrounding the Fermi surface. Notion of Fermi surface may be extended to the non-homogeneous systems using Wigner–Weyl calculus. The Fermi surface becomes coordinate dependent, it may be defined as the position of the singularity in momentum space of the Wigner transformed Green function. Then the topological invariant responsible for the stability of this Fermi surface is given by the same expression as for the homogeneous case, in which the Green function is replaced by its Wigner transformation while the ordinary products are replaced by the Moyal products. We illustrate the proposed construction by the examples of the systems, in which the given topological invariant is nontrivial and may be calculated explicitly.

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The topological stability of Fermi points is protected by expression [1, 2]

$$N_3 = \frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}\int_S dS^\mu G\partial^\nu G^{-1}G\partial^\rho G^{-1}G\partial^\lambda G^{-1}. \quad (1)$$

Here S is the surface that surrounds the given Fermi point, $G(p)$ is the Green function in momentum space. If chemical potential deviates from the position of the Fermi point, the latter is transformed to a Fermi surface. In this situation Eq. (1) protects its stability as a whole: it can change its form, and even shrink to a point, but it cannot disappear completely without a phase transition.

If in addition to the interations the inhomogeneity is present in the system (in particular, due to the presence of disorder or external magnetic field). Eq. (1) is undefined. In [3, 4] it was shown that expression similar to that of Eq. (1) is responsible for the chiral separation effect (CSE) in $3+1$D relativistic systems and in the condensed matter systems with emergent relativistic invariance (i.e. the conductivity of the latter is proportional to it):

$$N_3 = \frac{1}{48\pi^2}\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}\,\gamma^5\int_S dS^\mu G\partial^\nu G^{-1} G\partial^\rho G^{-1}G\partial^\lambda G^{-1}. \quad (2)$$

The further consideration of the CSE in the non-homogeneous systems [5–7] has led to the conclusion that Eq. (2) is to be extended further:

$$N_3 = \frac{1}{48\pi^2|V|}\epsilon_{\mu\nu\rho\lambda}\mathrm{tr}\,\gamma^5\int d^3x\int_S dS^\mu G_W\star\partial^\nu Q_W\star G_W \star\partial^\rho Q_W\star G_W\star\partial^\lambda Q_W. \quad (3)$$

Here $|V|$ is the three-volume of the system. It is assumed that γ^5 commutes or anticommutes with Q_W and G_W along the Fermi surface. (Below we omit ubscript W for brevity.) It is important to notice that in Eq. (3) surface S must have the form of the two hyperplanes $p_4 = \pm\epsilon \to 0$ situated just below the position of Fermi energy and just above it. It was noticed in [5–7] that these two hyperplanes may be safely deformed to the hypersurfaces of the other form surrounding the singularities of expression standing inside the integral over p. *The latter singularities actually represent the position of the Fermi surface depending on x. This is the way to extend the notion of the Fermi surface to the non-homogeneous systems.* However, the mentioned above deformation of hypersurface S should be accompanied by a modification of expressions for G_W and $\star$ (which was not mentioned in [5–7]). We already discussed this modification in [8]. In the present paper we touch this issue in more details.

Let us consider the situation, when Dirac operator $\hat{Q}$ commutes with Hermitian matrix $\hat{T}$. Noether current corresponding to this symmetry is denoted by $J^{(T)}$. Its

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response to external magnetic field is given by the integrals over the hypersurfaces $p_4 = \pm\epsilon$ that enclose the poles of the expression standing inside the integral:

$$\frac{\partial}{\partial\mu}\langle J_i^{(T,2)}\rangle = \sigma^{(T)} B_i. \qquad (4)$$

Here $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ is the magnetic field, averaging on the left hand side is over the whole system volume, while

$$\sigma^{(T)} = \frac{N_3^{(T)}}{4\pi^2} \qquad (5)$$

and

$$N_3^{(T)} = -\frac{1}{V\,24\pi^2}\sum_{p_4=\pm\epsilon}\int d^3x\int_{\Sigma_{p_4}} \mathrm{tr}_D\left(\hat{T}G^{(0)}\star dQ\star\wedge dG^{(0)}\star\wedge dQ\right). \qquad (6)$$

(Orientations of $\Sigma_{\pm\epsilon}$ are opposite.) The above expression is a topological invariant due to the presence of integration over the three-dimensional momenta $\mathbf{p}$ and coordinates $\mathbf{x}$. Here $Q(x,p) = Q_W(x,p)\Big|_{A=0}$ is Weyl symbol of Dirac operator at vanishihg external electromagnetic field, while $G^{(0)}\star Q = 1$.

There exists also the representation of Eq. (6) in the form of an integral over the infinite hypersurface of more general form embracing the singularities of expression standing in the integral in Eq. (6).

In particular, we can represent

$$N_3^{(T)} = -\frac{1}{V\,24\pi^2}\int d^3x\int_{\Sigma(\mathbf{x})}\mathrm{tr}_D\Big(\hat{T}\tilde{G}^{(0)}\star d\tilde{Q}\star\wedge\, d\tilde{G}^{(0)}\star\wedge d\tilde{Q}\Big). \qquad (7)$$

In transition from Eq. (6) to this expression we perform a smooth modification of Dirac operator:

$$Q(x,(\mathbf{p},\pm\epsilon)) \to Q(x,(\mathbf{p},\pm\omega_{\pm}(\mathbf{p},\mathbf{x}))) \equiv \tilde{Q}(\mathbf{x},\mathbf{p}).$$

Here $\omega_{\pm}(\mathbf{p},\mathbf{x})$ is a function, which describes the dependence of p_4 on $\mathbf{p}$ on the hypersurface $\Sigma(\mathbf{x})$. The latter hypersurface in momentum space for any $\mathbf{x}$ embraces the singularities of the expression standing inside the integral. For the case of a homogeneous system these singularities appear along the Fermi surface (or Fermi point). In the inhomogeneous case the geometrical place in momentum space of such singularities depends on $\mathbf{x}$ and generalizes the notion of the Fermi surface. We will refer to these surfaces in momentum space as to the coordinate depending Fermi surfaces. In turn, in transition from Eq. (6) we deform also the Green function as $G^{(0)}(x,(\mathbf{p},\pm\epsilon)) \to \tilde{G}(\mathbf{x},\mathbf{p})$, where the latter function obeys the *transformed* Groenewold equation

$$\tilde{Q}(\mathbf{x},\mathbf{p})e^{\frac{i}{2}(\overleftarrow{\partial}_{\mathbf{x}}\overrightarrow{\partial}_{\mathbf{p}}-\overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_{\mathbf{x}})}\tilde{G}^{(0)}(\mathbf{p},\mathbf{x}) = 1. \qquad (8)$$

The given smooth modification $Q \to \tilde{Q}$ and $G^{(0)} \to \tilde{G}^{(0)}$ does not result in a jump of the value of N_3 because defined this way it is the topological invariant.

Dirac operator Q is regular, but the Groenewold equation determines $G^{(0)}$, which may have singularities. We assume that all these singularities are situated at $p_4 = 0$. Notice that the star product contains only derivatives with respect to spatial components of momenta. Therefore, we can calculate expression standing inside the integral in Eq. (6) within each hypersurface $p_4 = const$ separately. As a result it has a discontinuity accross the hyperplane $p_4 = 0$ because the value of $G^{(0)}$ just below the singularity differs from its value just above it. Suppose that at each x the distance in momentum space between the disconnected pieces of Fermi surface is much larger than the inverse correlation length. Then there exists the region between these pieces, where there is no jump in $G^{(0)}$ accross the hyperplane $p_4 = 0$, and there in Eq. (6) the pieces of Σ_ϵ and $\Sigma_{-\epsilon}$ cancel each other at $\epsilon \to 0$. Then for any $\mathbf{x}$ the hypersurface $\Sigma = \Sigma_\epsilon \bigcup \Sigma_{-\epsilon} = \bigcup_i \Sigma^{(i)}$ at $\epsilon \to 0$ may be divided into the hypersurfaces $\Sigma^{(i)}$ encompassing the disconnected Fermi surfaces/Fermi points.

In the presence of proper ultraviolet regularization the Bloch theorem (absence of total electric current in equilibrium) in line with the Nielsen–Ninomiya theorem (absence of a chiral fermion in lattice regularization) gives rise to the conclusion that $N_3^{(T)} = 0$ for $\hat{T} = 1$. However, the values of

$$N_3^{(T,i)} = -\frac{1}{V\,24\pi^2}\int d^3x\int_{\Sigma^{(i)}(\mathbf{x})}\mathrm{tr}_D\Big(\hat{T}\tilde{G}^{(0)}\star d\tilde{Q}\star\wedge\, d\tilde{G}^{(0)}\star\wedge d\tilde{Q}\Big) \qquad (9)$$

may remain nonzero not only for $\hat{T} \neq 1$. If tuning the Fermi level we are able to reduce the i-th Fermi surface to a point, we are speaking of the ($\mathbf{x}$ depending) Fermi point.

In this case Eq. (9) is a topological invariant (protected by symmetry $\hat{T}$) responsible for the stability of the Fermi point. Modifying the system smoothly one cannot change its value until the two Fermi points/Fermi surfaces meet each other and annihilate. The latter may occur if the two corresponding values of $N_3^{(T,i)}$ are equal in absolute values and opposite in signs.

It is instructive to represent Eq. (7) in an arbitrary parametrization of the hypersurface $\Sigma^{(i)}$ by real numbers $\mathbf{k} = (k_1,k_2,k_3)$. We still assume that the positions of Fermi surfaces/Fermi points depend on $\mathbf{x}$. However, we suppose that for any $\mathbf{x}$ exists the common hypersurface in momentum space that does not depend on $\mathbf{x}$ but surrounds the given Fermi surface for *any* $\mathbf{x}$. This means that the corresponding function $\omega_\pm(\mathbf{p})$ depends only on $\mathbf{p}$. We express $\mathbf{p}(\mathbf{k})$ through the new parameters $\mathbf{k}$. In terms of the latter the necessary representation reads

$$N_3^T = \frac{\epsilon_{ijk}}{24\pi^2}\int_\Sigma d^3k \int \frac{d^3\xi}{V}\mathrm{tr}\Big[\hat{T}\tilde{G}\circ\partial_{k_i}\tilde{Q}\circ \tilde{G}\circ\partial_{k_j}\tilde{Q}\circ\tilde{G}\circ\partial_{k_k}\tilde{Q}\Big] \quad (10)$$

with

$$\circ = e^{\frac{i}{2}\left(\overleftarrow{\partial_{\xi^i}}\overrightarrow{\partial_{k_i}}-\overleftarrow{\partial_{k_i}}\overrightarrow{\partial_{\xi^i}}\right)} \quad (11)$$

and $V=\int d^3\xi$. Correspondingly, the limit of infinitely large V in Eq. (10) is to be considered. Vector $\boldsymbol{\xi}$ represents the new parametrization of coordinate space. These new coordinates are determined by equation $\frac{\partial \xi^i(\mathbf{k},\mathbf{x})}{\partial x^j} = \frac{\partial p_j(\mathbf{k})}{\partial k_i}$. Therefore, $\xi^i = \frac{\partial p_j(\mathbf{k})}{\partial k_i}x^j$ $(i,j = 1,2,3)$. Correspondingly, $\int d^3\xi = \mathrm{Det}\,\frac{\partial p(\mathbf{k})}{\partial k}\int d^3x$.

By $\tilde{Q}$ in Eq. (10) we understand

$$\tilde{Q}(\mathbf{p}(\mathbf{k}),\mathbf{x}(\mathbf{k},\boldsymbol{\xi})) \equiv Q((\mathbf{p}(\mathbf{k}),\pm\omega_\pm(\mathbf{p}(\mathbf{k}))), x(\mathbf{k},\boldsymbol{\xi})) \quad (12)$$

with $x^j = (\frac{\partial k_k}{\partial p_j}\xi^k)$ $(i,j=1,2,3)$. Here there is no dependence on x^4 because we consider the equilibrium system, while $p = (\mathbf{p}(\mathbf{k}),\pm\omega_\pm(\mathbf{k}))$.

By $\tilde{G}$ we then understand the function inverse to $\tilde{Q}$ with respect to the $\circ$ product:

$$\tilde{Q}\circ\tilde{G} = 1. \quad (13)$$

One can see that Eq. (10) is reparametrization – invariant. Namely, it is invariant under the transformation

$$k_i \to \tilde{k}_i(k), \quad \xi \to \tilde{\xi}^j = \frac{\partial k_i}{\partial \tilde{k}_j}\xi^i \quad (14)$$

One can check easily that the $\circ$ product of Eq. (11) is equivalent to the $\star$ product of Eq. (8). Transition between the two is given by the reparametrization transformation $x\to\xi, p\to k$. Recall that in Eq. (8) the differentiation with respect to $\mathbf{p}$ entering the Moyal product takes into account the dependence of p_4 on $\mathbf{p}$. The particular case when $\boldsymbol{\xi}$ coincides with $\mathbf{x}$ while $\mathbf{k}$ coincides with $\mathbf{p}$ brings us back to Eq. (7) (in the particular case when Σ does not depend on x).

We illustrate the proposed construction by the examples of the systems, in which the given topological invariant is nontrivial and may be calculated explicitly.

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