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A PATH-INTEGRAL QUANTIZATION OF THE STRAIGHT-LINE STRING

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A path-integral quantization of the relativistic straight-line string is proposed. In an explicitly covariant form starting with the initial four-dimentional dynamics of the relative coordinate r_μ the three-dimentional one is derived. A connection between constraints appearing in the canonical formalism and the path integral quantization is discussed.

1. *Introduction.* Various kinds of string-like models have been employed to describe the world of hadrons [1]. The massless relativistic straight-line string is usually considered as the simplest dynamical basis of the model of hadrons. This model has been quantized in the canonical quantization formalism [2]. Dynamics of quark and gluon fields leads us to the study of dynamics of the relativistic string with masses at the ends [3], in which the quarks carry a finite fraction of the energy-momentum of the hadron. But even the simplest version of the straight-line string with masses at the ends cannot be made tractable in the canonical quantization formalism [4]. In this paper we propose a path-integral Lorentz-covariant approach to the quantization of the massless relativistic straight-line string which can be generalized to the case of the straight-line string with masses at the ends [6].

2. *Gaussian representation for the action of the straight-line string.* The standard form of the action of the straight-line string in Euclidean space is

$$S = \sigma_0 \int_0^1 d\gamma \int_0^1 d\beta [\dot{w}^2 \cdot w'^2 - (\dot{w} \cdot w')]^{1/2} \tag{1}$$

where $w_\mu(\gamma,\beta)$ are the coordinates of the string world surface

$$w_\mu(\gamma,\beta) = z_\mu(\gamma) \cdot \beta + \bar{z}_\mu(\gamma) \cdot (1-\beta) \tag{2}$$

and $z_\mu(\gamma), \bar{z}_\mu(\gamma)$ are the coordinates of the ends of the string. The dot and the prime stand for the derivatives over γ and β parameters. Therefore we are to consider and make it tractable the following functional integral in the Euclidean space

$$G = \int Dz_\mu(\gamma) D\bar{z}_\mu(\gamma) \exp[-S]. \tag{3}$$

The action is invariant under the reparametrization

$$\gamma \to f(\gamma,\beta), \quad w_\mu(\gamma,\beta) \to w_\mu(f(\gamma,\beta),\beta) \tag{4}$$

with the function $f(\gamma,\beta)$ satisfying the conditions

$$f(0,\beta) = 0, f(1,\beta) = 1, \frac{\partial f(\gamma,\beta)}{\partial \gamma} > 0 \,. \tag{5}$$

It is convenient to introduce a "center of mass" coordinate $R_\mu(\gamma)$ and a relative coordinate $r_\mu(\gamma)$ as follows [3]

$$R_\mu(\gamma) = \frac{1}{2}(\bar{z}_\mu(\gamma) + z_\mu(\gamma)) \quad , r_\mu(\gamma) = z_\mu(\gamma) - \bar{z}_\mu(\gamma) \tag{6}$$

so that

$$w_\mu = R_\mu + (\beta - 1/2) \cdot r_\mu. \tag{7}$$

The boundary conditions can be imposed in the Lorentz-invariant way

$$R_\mu(1) - R_\mu(0) = T \cdot u_\mu, u_\mu \cdot u^\mu = 1 \tag{8}$$

and $r_\mu(0), r_\mu(1)$ are also fixed.

To develop a procedure to evaluate path-integral (3) we use the auxiliary fields formalism, as is usually done in the string theory [5].

Let us rewrite (3) as

$$G = \int Dr \ \ DR \ \ Dh_{ab} \exp[-\sigma_0 \int \sqrt{h} d^2\xi] \delta(\partial_a w_\mu \partial_b w^\mu - h_{ab}(\xi)) =$$

$$= \int Dr \ \ DR \ \ Dh_{ab} \int_{-i\infty}^{+i\infty} D\lambda^{ab} \exp[-\sigma_0 \int \sqrt{h} d^2\xi] \times$$

$$\times \exp[+ \int \sqrt{h} \lambda^{ab} h_{ab} d^2\xi] \exp[- \int \sqrt{h} \lambda^{ab} \partial_a w_\mu \partial_b w^\mu d^2\xi] \tag{9}$$

where $d^2\xi = d\gamma d\beta$, $\xi_1 = \gamma$, $\xi_2 = \beta$, $h \equiv det h$.

It is convenient to decompose [5]

$$\lambda^{ab}(\xi) = \alpha(\xi) h^{ab}(\xi) + f^{ab}(\xi) \tag{10}$$

with

$$f^{ab} h_{ab} = 0, h^{ab} \equiv (h^{-1})^{ab}. \tag{11}$$

In a similar way as it has been done in the case of Nambu-Goto string [5] we can prove that in the continuum limit $\alpha(\xi)$ and $f^{ab}(\xi)$ can be replaced by their mean values

$$< \alpha(\xi) > \to \bar{\alpha} , \quad < f^{ab}(\xi) > \to 0. \tag{12}$$

Equation (12) reflects the fact, that $\alpha(\xi)$ is a scalar, while $f^{ab}(\xi)$ is a traceless tenzor.

Using (12) we obtain the following expression for G:

$$G = \int Dr \ \ DR \ \ Dh_{ab} \ \exp[-(\sigma_0 - 2\bar{\alpha}) \int \sqrt{h} d^2\xi] \ \exp[-\bar{\alpha} \int \sqrt{h} h^{ab} \partial_a w_\mu \partial_b w^\mu d^2\xi] \tag{13}$$

with the new action which is quadratic in w_μ and contains the new auxiliary fields h_{ab}.

3. *Integration over the auxiliary fields.* The invariance (4) makes it convenient to introduce the new variables $\tilde{\nu}(\beta)$, $f(\xi)$, $\eta(\xi)$. Separating out the collective mode $\tilde{\nu}(\beta)$ and the field $f(\gamma, \beta)$, satisfying conditions (5), we have

$$\hbar \equiv \frac{h}{h_{22}^2} = (T\sigma\tilde{\nu}(\beta))^2 (\frac{\partial f(\gamma, \beta)}{\partial \gamma})^2. \tag{14}$$

And making a simple rescaling of $\hbar_{12}$ we also introduce the variable $\eta(\gamma,\beta)$ instead of h_{12}

$$\hbar_{12} \equiv \frac{h_{12}}{h_{22}} = (\frac{\partial f(\gamma,\beta)}{\partial \gamma})(T\eta(\gamma,\beta)) \tag{15}$$

where T enters boundary condition (8). Taking into account the fact, that

$$Dh_{11}\ Dh_{22}\ Dh_{12} = D\hbar\ D\hbar_{12}\ h_{22}^2\ Dh_{22} \tag{16}$$

and using the well known in the string theory formula [5]

$$D\hbar \sim \exp[-\frac{\text{const}}{\epsilon}\int \sqrt{\hbar} d^2\xi] D\tilde{\nu}(\beta) Df(\gamma,\beta) \tag{17}$$

where $1/\epsilon \sim \Lambda$ is the ultraviolet cut-off scale, we arrive at the following expression after changing the integration over $d\gamma$ by $Tdf(\gamma,\beta) \equiv d\tau$ and gaussian integration over $h_{22} \geq 0$

$$G = \int DR_\mu \quad Dr_\mu \quad D\tilde{\nu}(\tau,\beta) d\eta(\tau,\beta) \exp[-A] \tag{18}$$

where

$$A = \int_0^T d\tau \int_0^1 d\beta \frac{1}{2\tilde{\nu}}[\dot{w}^2 + (\sigma\tilde{\nu})^2 r^2 - 2\eta(\dot{w}r) + \eta^2 r^2] \tag{19}$$

and trivial rescaling $z, \bar{z} \rightarrow (\frac{\sigma}{2\bar{\alpha}})^{1/2} z, (\frac{\sigma}{2\bar{\alpha}})^{1/2} \bar{z}$ has been done.

At first we notice that the action doesn't depend on $f(\gamma,\beta)$ which reflects the invariance (4). So that the integral over $Df(\tau,\beta)$ can be factored out and it is equal to the volume of the reparametrization group.

In the standard way [5] we have introduced the physical quantity σ, which entered expression (14) and (19)

$$\sigma^2 = \bar{\alpha}(\sigma_0 - 2\bar{\alpha} + \frac{\text{const}}{\epsilon}). \tag{20}$$

In the action the function $\eta(\tau,\beta)$ is integrated over β being multiplied by function $\tilde{\nu}(\beta)$. In what follows the integration over $\tilde{\nu}$ will be performed by the steepest descent method and in the extremum $\tilde{\nu}((\beta - \frac{1}{2})) = \tilde{\nu}(-(\beta - \frac{1}{2}))$. So we can consider only the class of functions $\tilde{\nu}((\beta - 1/2)^2)$, which are even functions of $(\beta - 1/2)$.

It is convenient to decompose the functions $\eta(\tau,\beta)$ in orthogonal polinomials $P_n(\beta)$ with weight $\nu(\beta) \equiv 1/\tilde{\nu}(\beta)$

$$\eta(\tau,\beta) = \sum_n P_n(\beta) k_n(\tau) \tag{21}$$

$$\int_0^1 d\beta \nu(\beta) P_n(\beta) P_m(\beta) = \delta_{mn}. \tag{22}$$

After gaussian integration over $R(\gamma)$ with the condition (8)

$$\int DR \rightarrow \int D\dot{R} \int_{-i\infty}^{i\infty} d^4\lambda \exp[\int_0^T \lambda^\mu (\dot{R}_\mu - u_\mu) d\tau] \tag{23}$$

it is easy to prove that the action can be represented, with λ being rewritten as $i\lambda$, in the following form

$$S = \frac{1}{2}\int_0^T d\tau[a_3\dot{r}^2 + \sigma^2 a_{-1}r^2 + r^2\sum_{n=1}^{\infty}k_n^2(\tau) - 2i(a_1)^{-1/2}k_0(\tau)(\lambda r) -$$

$$-2(a_3)^{1/2}(\dot{r}r)k_1(\tau) + \frac{\lambda^2}{a_1} + 2i(\lambda u)] \tag{24}$$

where we have introduced the following notation

$$a_1 = \int_0^1 \nu d\beta, \quad a_3 = \int_0^1 (\beta - 1/2)^2 \nu d\beta, a_{-1} = \int_0^1 \frac{d\beta}{\nu} \ . \tag{25}$$

The function $k_0(\tau)$ enters only the fourth term in this expression and the integration over $Dk_0(\tau) = \prod_{i=1}^N dk_0(\tau_i)$, with N tending to infinity, gives the factor proportional to the infinite product of δ-functions;

$$\prod_{i=1}^{\infty}\delta(\lambda r(\tau_i)). \tag{26}$$

This means that there is a dynamical condition

$$(\lambda r) = 0 \, . \tag{27}$$

Integrations over $d^4\lambda$ and Dk_n with $n \geq 1$ lead (effectively in the limit $T \to \infty$) to the following expression (up to a change of the measure)

$$G = \int d\nu(\beta) Dr_\mu \delta(ru)\exp[-\frac{1}{2}\int_0^T d\tau[a_1 + (\dot{r}^2 - \frac{(r\dot{r})}{r^2})a_3 + \sigma^2 a_{-1}r^2] \tag{28}$$

where we have used the fact that the extremum value of λ_μ is

$$\lambda_\mu \sim u_\mu. \tag{29}$$

It is important that there are two constraints

$$(ru) \sim (rP) = 0 \tag{30}$$

$$(rp) = 0 \tag{31}$$

where P_μ is the total momentum of the string and

$$p_\mu = a_3(\dot{r}_\mu - \frac{(\dot{r}r)r_\mu}{r^2}) \tag{32}$$

is the relative momentum of the string.

The first constraint (30) means that only transverse to the total momentum P_μ components of r_μ are responsible for the dynamics of the string. The second one (31) reflects the fact that the action doesn't depend on the components of p_μ longitudinal to r_μ.

Let consider the rest system of the meson $u_\mu = (1, \vec{0})$ and go over from the Euclidean to the Minkovsky space

$$d\tau_E \rightarrow i d\tau_M \tag{33}$$

The Hamiltonian coresponding to the action (28) is

$$H(\vec{p}, \vec{r}) = 1/2\{\frac{1}{a_3}\frac{\hat{L}^2}{\vec{r}^2} + \sigma^2 a_{-1}\vec{r}^2 + a_1\} \tag{34}$$

where $\hat{\vec{L}} = (\vec{r} \times \vec{p})$ is the operator of the angular momentum.

Since the hamiltonian does not contain the radial part of the kinetic term the field $\vec{r}^2$ is not a dynamical one. Thus in the spirit of the canonical formalism we must exclude the field $\vec{r}^2$. This can be done by solving its equation of motion for a fixed value of the orbital momentum l

$$-\frac{l(l+1)}{a_3\vec{r}^4} + \sigma^2 \cdot a_{-1} = 0. \tag{35}$$

Inserting this extremum value of $\vec{r}^2$ into expression (34) we arrive at the final expression for the hamiltonian

$$H(\nu, l) = \frac{1}{2}a_1 + \sigma\sqrt{a_{-1}/a_3}\sqrt{l(l+1)}. \tag{36}$$

Solving eq.(36) for the extremum of $\nu(\beta)$ with the conditions (25) we find

$$\nu(\beta) = (\frac{8\sigma\sqrt{(l+1)l}}{\pi})^{1/2}\frac{1}{\sqrt{1-4(\beta-1/2)^2}} \tag{37}$$

with $\nu(\beta)$ playing the role of the energy density of the string. This solution corresponds to the spectrum of the hamiltonian

$$E_l^2 = M_l^2 = 2\pi\sigma\sqrt{l(l+1)} \tag{38}$$

which agrees with the result obtained for the straight-line string in the canonical quantization formalism [2].

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