FROM INSTANTONS TO INFLATIONARY UNIVERSE

I.M.Khalatnikov[*] [+], *P.Schiller*[+]

[*]*L.D.Landau Institute for Theoretical Physics*
Russian Academy of Sciences
117940 Moscow, Kosygin Street 2, Russia

[+]*Max-Planck-Institut für Astrophysik*
Karl-Schwarzschild-Str. 1
8046 Garching bei München, Germany

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The present paper is based on a theory which includes gravity and a complex scalar field [1]. It is shown shown in such a theory we can proceed the evolution from instantons in the classically forbidden (Euclidean) region in minisuperspace to the inflationary universe in the classically allowed (Minkowski) region. The characteristics for the Hamilton-Jacobi equation, which define the action in the quasiclassical approximation, are described by four differential equations of first order. This four dimensional dynamical system was integrated numerically. In the Euclidean closed region we found two types of instantons. It is shown that the instantons correspond to extremal trajectories. The existence of two types of instantons gives different possibilities for tunneling from Euclidean to Minkowski regions and for creation of inflationary universes.

The well known contemporary theory of an inflationary universe is based on a Friedmann homogeneous model in the presence of a real scalar field. In such a model it is possible to analyse all possible inflationary solutions for three types of geometry (flat, open and closed) depending on the inital conditions, which we fix on some surface in phase-space and which we define as the end of the quantum era [2]. At this initial surface the density of energy has an order of the Planck energy $\epsilon_p \sim m_p^4$.

The quantum creation of an universe in such model was studied in [3]. The only result which was possible to get was the creation of a large, but empty universe (free from field) which as it is known [2] cannot inflate. In [1] we formulated a model which considers gravity (Friedmann closed model) induced by complex scalar field. One of the reasons to consider a complex instead of real scalar field is the fact, that the energy-momentum tensor for such a field simply corresponds with the hydrodynamical energy-momentum tensor usually used in general relativity. The main result obtained in [1] is the existence of a closed area in minisuperspace (metric and field) which is classically forbidden and which has a boundary partly convex and partly concave. Due to this fact there are two possible types of instantons, which are able after tunneling to the classically allowed region to create the inflationary universe. The model of complex scalar field coupled to gravity was previously applied by K.Lee [4] to study wormhole physics, a problem different from ours.[1)]

Let as recall the main equations of our model [1]. The action for gravitation and a complex massive scalar field has the following form [2)]

$$S = \int d^4x\sqrt{-g}\left(-\frac{M_p^2}{16\pi}R + \frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi^*_{,\nu} - \frac{1}{2}m^2\varphi\varphi^* - \lambda\right), \tag{1}$$

where the metric $g_{\mu\nu}$ for a homogeneous closed model and the field φ we choose as follows

$$\varphi = \sqrt{\frac{3}{4\pi}}M_p x(t)e^{i\Theta(t)}, \tag{2}$$

$$g_{\mu\nu} = \mathrm{diag}\left(\frac{N^2(t)}{m^2}; -\left(\frac{2}{3\pi M_p^2 m}\right)^{2/3} e^{2\alpha(t)}h_{ij}\right), \tag{3}$$

where $a(t) = \left(\frac{2}{2\pi M_p^2 m}\right)^{1/3} e^{\alpha(t)}$ is the scale factor of the space section (3-sphere) and h_{ij} is the metric of the unit 3-sphere.

The action (3) includes also the cosmological term λ. After standard calculations we obtain the Hamiltonian form of the action

$$S_H = \int dt(p_i\dot{q}^i - L) = \int dtN\left[-\frac{e^{-3\alpha}}{2}p_\alpha^2 + \frac{e^{-3\alpha}}{2}p_x^2 + \frac{e^{-3\alpha}}{2x^2}p_\Theta^2 - \right.$$

$$\left. -\frac{1}{2}\gamma e^\alpha + \frac{1}{2}e^{3\alpha}(x^2+\Lambda)\right], \tag{4}$$

where the canonical momentums are

$$p_\alpha = -\frac{e^{3\alpha}}{N}\dot{\alpha}; \quad p_x = \frac{e^{3\alpha}}{N}\dot{x}; \quad p_\Theta = \frac{e^{3\alpha}x^2}{N}\dot{\Theta}. \tag{5}$$

We introduce a dimensionless cosmological constant Λ and $\gamma = \left(\frac{3\pi M_p^2}{2m^2}\right)$.

Using the conservation of current j_μ of the field φ we can exclude the constant momentum $p_\Theta = Q$, where Q is a new constant of the theory playing a crucial role for the results. The final expression for the Hamiltonian action for the two-dimensional minisuperspace is (with the choise of lapse $N=1$):

$$S_H = \frac{1}{2}\int dt[e^{-3\alpha}(-p_\alpha^2 + p_x^2) + m^2], \tag{6}$$

[1)] Authors thank D.Brill for attracting their attention to the work [4]

[2)] We expect that the introdiction of interaction of the field of a type $\lambda\varphi^4$ or a Higgs field will not change qualitativelly the main results, but it needs special investigation.

where we introduce the notation

$$m^2(\alpha, x) = \frac{e^{-3\alpha} Q^2}{x^2} - \gamma e^{\alpha} + e^{3\alpha}(x^2 + \Lambda) \tag{7}$$

and the constraint equation is

$$H = e^{-3\alpha}(-p_\alpha^2 + p_x^2) + m^2(\alpha, x) = 0. \tag{8}$$

The Wheeler-DeWitt equation which corresponds to equation (8) ($p_k \to \frac{\hbar}{i}\nabla_k$) is

$$\left[\hbar^2 e^{-3\alpha}\left(\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial x^2}\right) + m^2(\alpha, x)\right]\Psi(\alpha, x) = 0. \tag{9}$$

This WD equation we write in the simplest form assuming that metric of the minisuperspace is

$$g_{\mu\nu} = \begin{bmatrix} e^{3\alpha} & 0 \\ 0 & -e^{3\alpha} \end{bmatrix} \tag{10}$$

and its commutation brackets with momentum p_α in the quasiclassical approximation will be not important. The obtained WD equation (9) is a Klein-Gordon equation with m^2 depending on metric α and field x and which can be negative in some part of the minisuperspace. Let us mention, that when we write the WD equation in the form (9) we asume, that the $p_\ominus$ variable is frozen and the corresponding motion is not quantized. Using the analogy between our variables α and x, and time and space coordinates, then one possible interpretation is that in the region $m^2 < 0$ (see [3]) the processes of creation of particles are occuring in the second quantized theory of fields of universes $\Psi(\alpha, x)$. This region is classically forbidden. The form of this region depends on three parameters: γ (of the mass of the scalar field), the constant Q and the cosmological constant Λ. The possible forms are given on fig.1. The remarkable fact, which we observe is the possibility to obtain not only convex boundaries of the regions $m^2 < 0$, but also partly convex and partly concave boundaries. The typical form of the surface m^2 as a function of α and x and the corresponding equipotential map are shown on fig.2.

The constant Q is responsible for a closing of the upper part of the region $m^2 < 0$ and the cosmological constant Λ for a closing of the right part. The number of created universes and the most probable trajectories we can find from the solution of a quasiclassical equations in classically allowed and forbidden regions.

1. Minkowski space - classically allowed region, $m^2 > 0$. The action in this case is

$$S_M = \frac{1}{2}\int dt[p^\mu p_\mu - m^2(\alpha, x)] \tag{11}$$

and the constraint equation is

$$H = \frac{1}{2}[p^\mu p_\mu + m^2(\alpha, x)] = 0. \tag{12}$$

Corresponding Hamilton-Jacobi equation is

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m^2 = 0 \tag{13}$$

and

$$p^x = \dot{x} \qquad g_{\mu\nu} = \begin{bmatrix} e^{3\alpha} & 0 \\ 0 & -e^{3\alpha} \end{bmatrix}, \qquad p_x = e^{3\alpha}\dot{x}$$

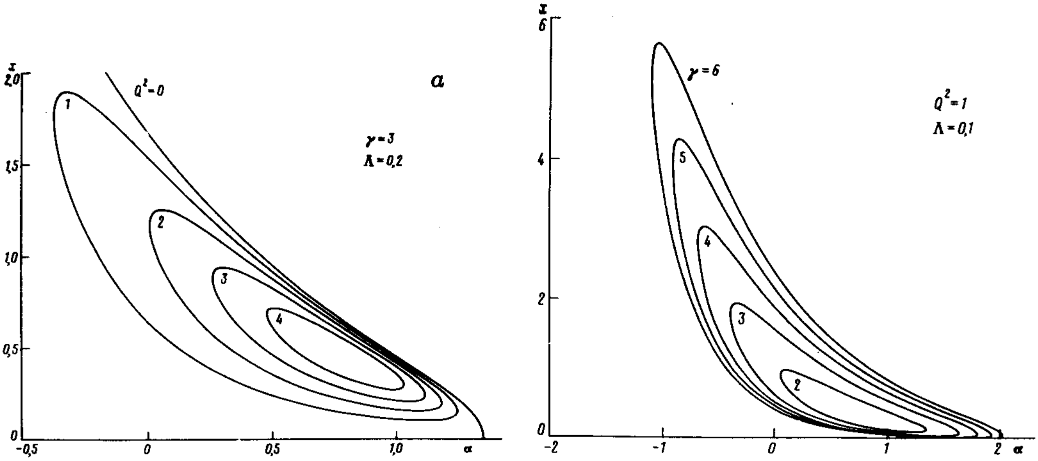

Fig.1. The boundary $m^2 = 0$: *(a)* for fixed $\Lambda = 0.2$ and $\gamma = 3$ and different $Q^2 = 0, 1, 2, 3, 4$, *(b)* for fixed $\Lambda = 0.1$ and $Q^2 = 1$ and different $\gamma = 2, 3, 4, 5, 6$

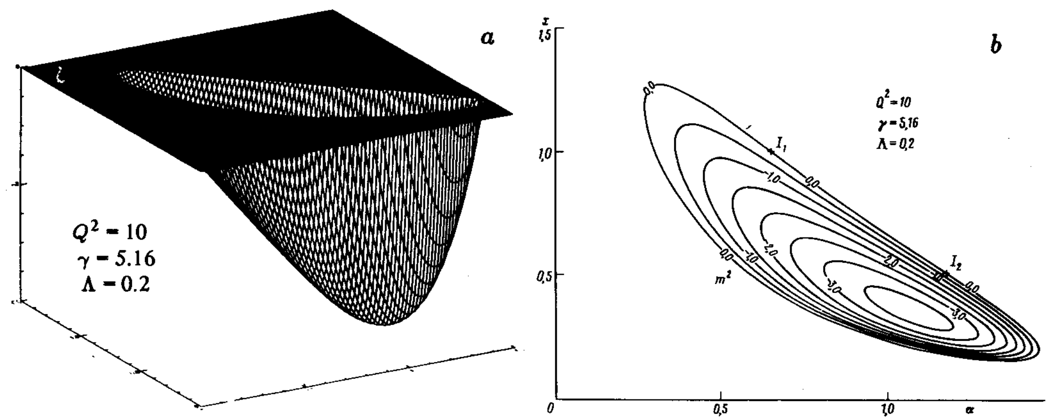

Fig.2. The typical form *(a)* and the equipotential map *(b)* of the surface m^2 inside the Euclidean region ($\Lambda = 0.2$, $Q^2 = 10$, $\gamma = 5.1578$)

$$p^\alpha = \dot{\alpha}, \qquad\qquad p_\alpha = -e^{3\alpha}\dot{\alpha}. \tag{14}$$

The variation of the action S_M

$$\delta S_M = -\frac{1}{2}\int dt\delta x^\mu \left(2\dot{p}_\mu + m^2(\log\sqrt{-g})\mu + \frac{\partial m^2}{\partial x^\mu}\right), \tag{15}$$

gives the equation for characteristics

$$\dot{p}_\mu = -\frac{1}{2\sqrt{-g}}\frac{\partial}{\partial x^\mu}(m^2\sqrt{-g}), \qquad \sqrt{-g} = e^{3\alpha}. \tag{16}$$

These equations describe a four-dimensional dynamical system

$$\dot{x} = y,$$

$$\dot{\alpha} = z,$$

$$\dot{y} = -3yz - x + \frac{Q^2}{x^3 e^{6\alpha}},$$

$$\dot{z} = -3z^2 + 3(x^2 + \Lambda) - \frac{2\gamma}{e^{2\alpha}}, \tag{17}$$

with the first integral of motion

$$e^{3\alpha}(-z^2 + y^2) + m^2 = 0. \tag{18}$$

2. **Euclidean space - classically forbidden region,** $m^2 < 0$. The Euclidean action S_E can be obtained by Wick rotation of the proper time $t \rightarrow -it$

$$S_E = \frac{1}{2}\int dt[p^\mu p_\mu + m^2(\alpha, x)] \tag{19}$$

with the constraint equation

$$H = \frac{1}{2}[p_\mu p^\mu - m^2(\alpha, x)] = 0. \tag{20}$$

The variation of S_E,

$$\delta S_E = -\frac{1}{2}\int dt \delta x^\mu \left(-2\dot{p}_\mu + m^2(\log\sqrt{-g})_\mu + \frac{\partial m^2}{\partial x^\mu}\right), \tag{21}$$

gives the equation for instanton

$$\dot{p}_\mu = -\frac{1}{2\sqrt{-g}}\frac{\partial}{\partial x^\mu}(m^2\sqrt{-g}), \qquad \sqrt{-g} = e^{3\alpha}. \tag{22}$$

The Hamilton-Jacobi equation for the action of the instanton in quasiclassical approximation $(I = S_E)$ is

$$g^{\mu\nu}\frac{\partial I}{\partial\alpha^\mu}\frac{\partial I}{\partial\alpha^\nu} - m^2 = 0 \tag{23}$$

Its characteristics describe a four-dimensional dynamical system

$$\dot{x} = y,$$

$$\dot{\alpha} = z,$$

$$\dot{y} = -3yz + x - \frac{Q^2}{x^3 e^{6\alpha}},$$

$$\dot{z} = -3z^2 - 3(x^2 + \Lambda) + \frac{2\gamma}{e^{2\alpha}}, \tag{24}$$

with the first integral of motion

$$e^{3\alpha}(z^2 - y^2) + m^2 = 0. \tag{25}$$

In the Euclidean region it is necessary to find extremal instanton trajectories. The condition that the trajectory is extremal can be written as

$$\begin{cases} p^\mu_{(i,f)} p_{\mu(i,f)} = 0 \\ p^\mu_{(i,f)} e^\mu = 0 \end{cases}, \tag{26}$$

where i, j denote the initial and final points of the trajectory.

In (26) the first condition follows from the fact, that the trajectory starts from the boundary $m^2 = 0$ and the second means that the action has an extremum with respect to variation of the initial and final points of the trajectory along the curve $m^2 = 0$. If e_μ is the tangential vector to the curve $m^2 - 0$ defined by

$$\epsilon_\mu n^\mu = 0, \tag{27}$$

thet

$$n_\mu = \frac{\partial}{\partial x^\mu}(m^2)\mid_{m^2=0} \tag{28}$$

is the vector normal to the curle $m^2 = 0$. As $\frac{\delta I_0}{\delta x^\mu} = p_\mu$ the extremum condition along the direction e_μ can be written

$$\frac{\delta I_0}{\delta x^\mu} e^\mu = p_{\mu(i,f)} e^\mu = 0, \tag{29}$$

which coincides the second equation in (26).

There are two possibilities to satisfy the condition (26), as was shown in [1]. At the convex part of the boundary we can satisfy (26) only if $p_\mu = 0$. But as we have shown the boundary $m^2 = 0$ can have concave part with inflection points. In the last case there is a possibility of existence of a nontrivial solition which correspondos to the entrance of a trajectory into region $m^2 < 0$ with nonzero velocity along the tangent. Such a solution, if it exists, has to satisfy the condition

$$p_\mu = \beta n_\mu, \tag{30}$$

where the parameter β defines the velocity with which the trajectory enters the region $m^2 < 0$. As the vector p_μ at the entrance point is a light-like vector $(p_\mu p^\mu = 0)$ the only possibility satisfy this condition is to choose the parameters Q, y, Λ in such a way, that at one of the two inflection points the normal vector will be light-like, i.e.

$$(n_\alpha)^2 - (n_x)^2 = 0, \tag{31}$$

i.e.will have a slope of 45° (and the curve $m^2 = 0$ at this point will have inclination -46°). The inflection poin we need to provide the possibility for the trajectory to enter and exit the region $m^2 \leq 0$. On the boundary of the Euclidean region $m^2 < 0$ there are two singular points of the equation (24). They are defined by the conditions

$$y = 0, \qquad x - \frac{Q^2 e^{-6\alpha}}{x^3} = 0. \tag{32}$$

They coincide with the leftmost and rightmost points of the region $m^2 = 0$, where

$$\frac{\partial \alpha}{\partial x} = 0, \qquad n_x = \frac{\partial m^2}{\partial x} = -\frac{2Q^2}{e^{3\alpha} x^3} + 2xe^{3\alpha} = 0. \tag{33}$$

The trajectories of the dynamical systems (17) and (24) in Minkowski and Euclidean spaces naturally cannot be found analytically. We present the first results of numerical simulations. We start with the search for instantons in the region $m^2 < 0$. Selecting the parameter β in (30) which defines the initial velocity of the instanton in the minisuperspace we find a trajectory starting at the inflection point I_1, which moves in the direction of the singularity S_R and reachs a point R_1, where the velocity $p_\mu = 0$. After the reflection from this point it returns back to the inflection poin I_1 from which it can tunnel to the Minkowski space and leave the region $m^2 < 0$. At the reflection point R_1 which is becoming closer to the singularity S_R with decreasing Λ, the half-instation can also tunnel and emit inflationary universe. All these possibilities can be seen on fig.3*a*, where we show the Minkowski trajectory, describing the quantum oscillating universe tunneling into the Euclidean region, then the trajectory of the instanton and the inflationary universe leaving the Euclidean region. The corresponding rate of inflation is shown on fig.3*b*. The reflected trajectory from point R_1, which leaves the Euclidean region at the point I_1, describes in quasiclassical language an unlimited contraction, i.e. a collapse. This trajectory asymptotically tends to knot at infinity with the slope $\dot{x} \simeq -\dot{\alpha}(p_\mu p^\mu \approx 0)$. The described picture has common features with the well known process of quantum tunneling in quantum mechanics. Because of the large absolute value of the negative action of the instanton we shall not go here into all details of the process of tunneling of the instanton. We hope to return to this problem later.

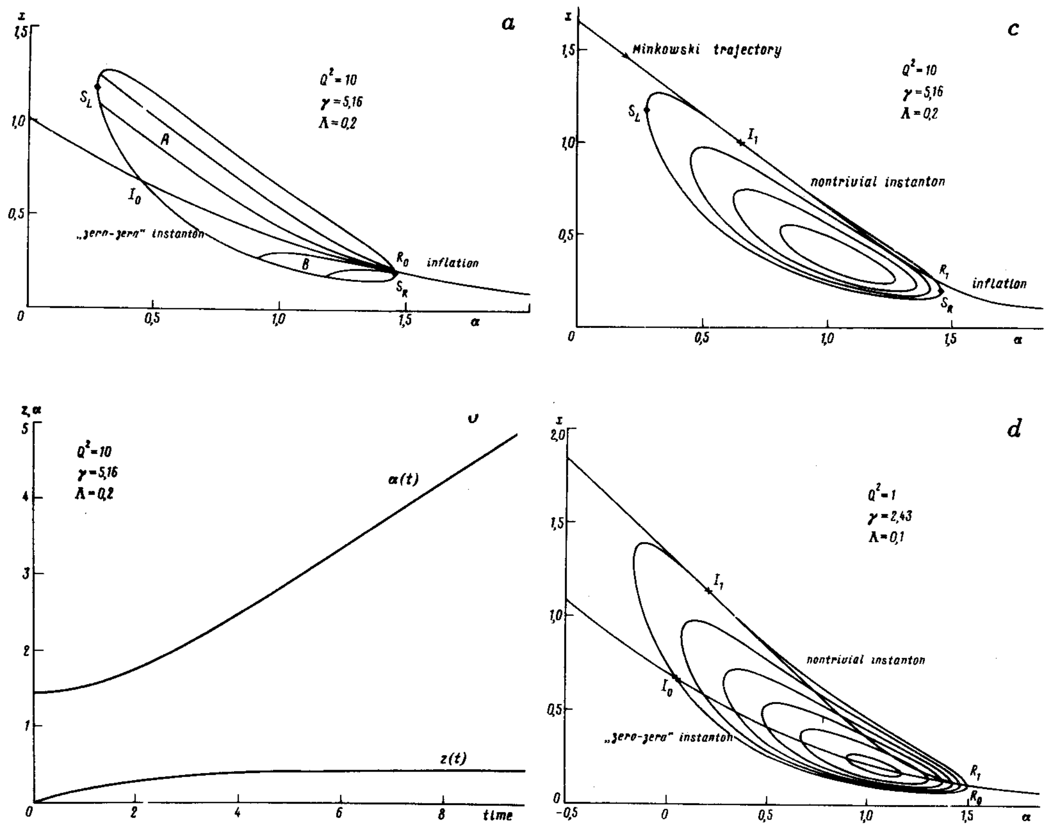

Fig.3. The "zero-zero" instation, the nontrivial instanton, the corresponding trajectories in Minkovski space for $\Lambda = 0.2$, $Q^2 = 10$, $\gamma = 5.1578$ *(a,c)*, $\Lambda = 0.1$, $Q^2 = 1$, $\gamma = 2.43$ *(d)* and the rate of the inflation *(b)*

Besides the described nontrivial instanton there is an second instanton ("zero-

zero" instanton) with a trajectory staring on the left convex part of the boundary at the point I_0 with $p_\mu = 0$ (see fig.3c) and propagating to the point $R_0(p_\mu = 0)$, where it can be either reflected or tunneled into the Minkowski region as an inflationary universe. The reflected half-instanton is returning to the point I_0, where it can be also reflected or tunneled into the Minkowski region. This trajectory will propagate in the direction of the above mentioned knot singularity at infinity. Let mention that on the concave part of the boundary of the Euclidean "banana-like" region there are two inflection points, but only one of them has a normal which is a null vector. Therefore inside this region we have simultaneouslly not more than two instantons: nontrivial or "zero-zero". The trajectories, which enter from Minkowski region into the Euclidean region through the points I_1 and I_0, are emitted from the repulsive knot located at $z = +\infty$.

There is also another interesting possibility that the "zero-zero" instanton can be many times reflected from the points I_0 and R_0 and in some sense be trapped in the Euclidean region. Such an instanton can be also considered as a source of "pair" creation of an inflationary universe and antiinflationary (collapsing) universe from "nothing" (see [5,6]).

The numerical analysis shows that the nontrivial instanton as well as the "zero-zero" instanton correspond to the local maximums of the action. Let us give the formula for the action of the instanton. From (19) and (20) we find

$$S_E = \int dtm^2 = \int dtp_\mu p^\mu = -\int_{\alpha_i}^{\alpha_f} d\alpha \sqrt{1-\left(\frac{\partial x}{\partial \alpha}\right)^2} \sqrt{-m^2 e^{3\alpha}}. \quad (34)$$

The numerical calculation of the action according to this equation for the choice of parameters Λ, γ and Q given at fig.3d gives $I_0 \simeq -7$ for the "zero-zero" instanton and $I_0 \simeq -4$ for the nontrivial instanton. The action of the DeSitter instanton in our notations is

$$I_{DS} = -\frac{\gamma^{3/2}}{3\Lambda}. \quad (35)$$

The absolute value of this action is always larger than the corresponding action in our Euclidean region. Let us emphasize and it is well known that the action for the gravitational instanton is negative. Therefore

$$|\Psi|^2 \sim e^{-2I} \quad (36)$$

for an instanton is a large number and it is not possible to give a probabilistic interpretation of this number. But if $|\Psi|^2$ is considered as a number of created universes in the region of nonstability $m^2 < 0$ that our Euclidean region can act as a amplifier, which selects only one of the trajectories from a large number arriving from the Minkowski region and makes the creation of the inflationary universe from the instanton most probable. This attracts attention also on the idea [3], that the ratio $e^{-2I_0}/e^{-2I_{DS}}$ can be interpreted as a probability of creation of the universe (for the definition of I_{DS} see (35)). The further analysis has to show how fruitful is our proposed model. In any case it gives us a bridge between the quantum and the classical cosmology. And if we consider the introduced parameters as world constants of a future theory, then our theory is a model which shows how the initial conditions for existing universe could follow from the theory and not to be implemented "ad-hoc".

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