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QUASIEQUILIBRIUM SOLUTION OF $1+d$ KPZ MODEL

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Properties of correlation functions of solutions of $1+d$ KPZ equation in the region of the strong interaction of fluctuations are considered. We prove that analytical continuation of the solution realizing at $d=1$ to the dimensions $1<d<2$ gives a solution with the dynamic index $z=(d+2)/2$. The possibility of alternative solutions is discussed.

Kardar - Parisi - Zhang (KPZ) equation is written as

$$\partial h/\partial t = \nu_0 \nabla^2 h + \lambda(\nabla h)^2 + \xi \,. \tag{1}$$

Here $h(t,\mathbf{r})$ is a scalar field, λ is an interaction constant, ν_0 is a diffusion coefficient and $\xi(t,\mathbf{r})$ is a random Gaussian "force" with the correlation function

$$\langle \xi(t_1,\mathbf{r}_1)\xi(t_2,\mathbf{r}_2)\rangle = 2T\nu_0\delta(t_1-t_2)\delta(\mathbf{r}_1-\mathbf{r}_2)\,, \tag{2}$$

where T is an effective temperature. Let us stress that a system described by (1) is far from equilibrium.

KPZ equation describes roughening of an interface in different cases, like grows of solids [1], two fluid flows [2,3], motion of domain walls [4] or boundaries of clusters [5] etc. This equation is equivalent to the Burgers equation [6-8]. It is also equivalent to the equation for the partition function of directed polymers [9] and of dislocations [10] or vortices [11] in a random potential (in these cases we should take the third coordinate instead of the time t). This variety of physical contexts is associated with the universal character of KPZ equation representing the long wavelength dynamics of any field h if it is invariant under $h \to h + \text{const}$ but not invariant under $h \to -h$.

At considering the interface in the $3d$ space or of the vortex in the $3d$ lattice the quantity h should be considered as a function of the $2d$ radius-vector $\mathbf{r}$. Then fluctuations of the field h are relevant. It appears that the case of "asymptotic freeedom" is realized that is the dimensionless coupling constant grows with increasing scale [12]. In this situation one can not say anything definite about the long wavelength properties of correlation functions of h on the basis of perturbative methods like renorm-group equations. Numerical experiments [13-15] show a scaling long wavelength behavior. From a theoretical point of view it is a surprising thing since in known exactly solvable models where "asymptotic freedom" is simulated the long-wavelength behavior of correlation functions is not of scaling type [16,17]. The possibility of the scaling behavior of the correlation functions of h is related to famous cancellations of ultraviolet divergences in KPZ model [18,19].

Therefore a problem appears to determine theoretically the dynamic index z characterizing correlation functions of h. As we know from the theory of second-order phase transitions it is useful to investigate the system with strong fluctuations not only in the physical dimensionality but also near it. The equation (1) may be considered in the space of any dimension d of $\mathbf{r}$ (the physical value being $d = 2$). We will investigate the behavior of the solutions of (1) for dimensions d between 1 and 2. Some conclusions concerning $d = 2$ can be derived further by extrapolation of obtained results. For $d = 1$ the exponent z can be derived exactly [12,4,20,21] it is $z = 1.5$ which is confirmed also numerically [13-15]. We will demonstrate that analytical continuation of the solution with $z = 1.5$ to the dimensions $1 < d < 2$ gives a solution with the exponent

$$z = \frac{d+2}{2}. \tag{3}$$

This solution may be called "quasiequilibrium" since it has the same index z as the equilibrium one.

To examine statistical properties of solutions of KPZ equation we will utilize a diagram technique of the type firstly developed by Wyld [22] for the problem of hydrodynamical turbulence and extended on a wide class of physical systems by Martin, Siggia and Rose [23]. A textbook description of the diagram technique can be found in the book by Ma [24]. Note that this diagram technique is a classical limit of the Keldysh diagram technique [25] applicable to any physical system. As it was demonstrated by de Dominicis [26] and Janssen [27] (see also [28] and [29]) Wyld's diagrammatic technique is generated by a conventional quantum field theory fashion starting from an effective action I. The corresponding methods of investigation can be found in the monograph by Popov [30].

The explicit expression for the effective action I can be constructed on the basis of nonlinear dynamic equations of a system. For KPZ equation the effective action is

$$I = \int dt\, d\mathbf{r} \left(\hat{h}\partial h/\partial t - \lambda \hat{h}(\nabla h)^2 + \nu_0 \nabla\hat{h}\nabla h + iT\nu_0\hat{h}^2 \right). \tag{4}$$

Here $\hat{h}$ is an auxiliary field "conjugated" to the field h. Introducing I enables us to express the correlation functions of h in terms of functional integrals. For example the pair correlation function is

$$F(t_1 - t_2, \mathbf{r}_1 - \mathbf{r}_2) = \langle h(t_1, \mathbf{r}_1) h(t_2, \mathbf{r}_2)\rangle \equiv \int \mathcal{D}h\mathcal{D}\hat{h} \exp(iI) h(t_1, \mathbf{r}_1) h(t_2, \mathbf{r}_2), \tag{5}$$

where the functional integration over the fields h and $\hat{h}$ is implied. It is also useful to define the Green's function

$$G(t_1 - t_2, \mathbf{r}_1 - \mathbf{r}_2) = -\langle h(t_1, \mathbf{r}_1)\hat{h}(t_2, \mathbf{r}_2)\rangle. \tag{6}$$

It is a susceptibility determining the linear response of the system to the external "force" to be added to the right-hand side of the equation (1). Namely G is an integral kernel in the linear relation between the external "force" and the average value of h. Therefore the value of $G(t, \mathbf{r})$ is equal to zero at $t < 0$ as a consequence of the causality principle.

Introduce instead of $h, \hat{h}$ new variables p, $\tilde{\mathrm{v}}$ in accordance with the definition

$$\hat{h} = \nabla_i p_i, \qquad \nabla_i h = -\tilde{v}_i + iTp_i, \tag{7}$$

where p_i and $\tilde{v}_i$ are potential fields that is

$$\nabla_i p_k = \nabla_k p_i\,, \qquad \nabla_i \tilde{v}_k = \nabla_k \tilde{v}_i\,. \tag{8}$$

These relations enable us to reduce the fields p_i and $\tilde{v}_i$ to gradients of scalar fields. The correlation functions $\langle \tilde{v}\, p\rangle$ and $\langle \tilde{v}\,\tilde{v}\rangle$ can be reduced to the functions G and F. The introduction of the new fields in accordance with (7) is analogous to one performed in the work [18] for the case $d=1$. For $d=1$ this new representation enabled us to prove some famous properties of the perturbation series generated by KPZ equation [18], it appears to be useful also for any dimension $1<d<2$.

In the new terms the effective action (4) is rewritten as the sum $I_0+I_1+I_2$ where

$$I_0 = \int dt\, d\mathbf{r}\, \Big(p_i \partial \tilde{v}_i/\partial t + \nu_0 \nabla_i p_i \nabla_k \tilde{v}_k\Big)\,, \tag{9}$$

$$I_1 = -\lambda \int dt\, d\mathbf{r}\, \Big(\nabla_i p_i \tilde{v}_k^2 + iT(2p_i p_k \nabla_i \tilde{v}_k - p_i^2 \nabla_k \tilde{v}_k)\Big)\,, \tag{10}$$

$$I_2 = \int dt\, d\mathbf{r}\, \lambda T^2 p_i^2 \nabla_k p_k\,. \tag{11}$$

The correlation functions $\langle \tilde{v}\, p\rangle$ and $\langle \tilde{v}\,\tilde{v}\rangle$ can now be calculated in the framework of the perturbation series. The bare values of G, D are determined by the second-order part I_0 of the effective action (9). Interaction vertices are determined by the third-order terms I_1 and I_2 in the effective action. The renormalized value of the Green's function $G=-\langle \tilde{v}_i p_i\rangle$ in Fourier representation can be written as

$$G(\omega,\mathbf{k}) = \Big(\omega + i\nu_0 k^2 - \Sigma(\omega,\mathbf{k})\Big)^{-1}\,, \tag{12}$$

where Σ is the self-energy function, represented by an infinite sum of one-particle irreducible diagrams.

Expressions for Σ and G in the region of the scaling behavior are

$$\Sigma(\omega,\mathbf{k}) = \mu_k \sigma(\omega/\mu_k)\,, \tag{13}$$

$$G(\omega,\mathbf{k}) = \Big(\omega - \mu_k \sigma(\omega/\mu_k)\Big)^{-1}\,, \tag{14}$$

where σ is a dimensionless function and

$$\mu_k \propto k^z\,, \tag{15}$$

z being the dynamic scaling exponent. The term $\nu_0 k^2$ in the region can be neglected.

Firstly we will prove that the "truncated" Green's function G_1 related to an average

$$\int \mathcal{D}\tilde{v}\, \mathcal{D}p\, \exp(iI_0 + I_1) v_i p_k \tag{16}$$

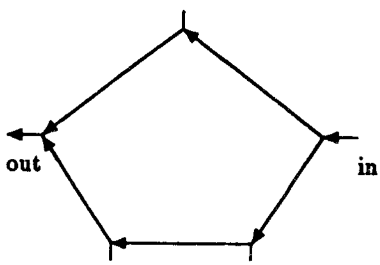
has the exponent (3). Then we will include the term I_2 (11) of the effective action and show that this inclusion will not change z. The idea of the proof is close to one proposed in the paper [18] where the case $d=1$ was examined.

The "truncated" Green's function G_1 can be written in the same form (14) as the complete function with the the self-energy function Σ_1 represented by an infinite sum of one-particle irreducible diagrams where only the vertices determined by (10) are present. We would like to stress that the "truncated" dressed correlation functions $\langle \tilde{v}_i \tilde{v}_k \rangle_1$ and $\langle p_i p_k \rangle_1$ are equal to zero. After a partial summation of diagrams we may arrive at a diagrammatic series for Σ_1 where the bare vertices but the dressed pair correlation functions figure. Since the correlation functions $\langle \tilde{v}_i \tilde{v}_k \rangle_1$ and $\langle pp \rangle_1$ are zero this series contains really the functions G_1 only.

Now we should include into consideration the omitted interaction term I_2 (11) in the effective action. Consider the "complete" self-energy function Σ which we will treat as a series over I_2. Each term in this series is actually determined by an infinite sequence of diagrams generated by the interaction terms in I_1 (10). We will imply the same partial summation of the diagrams dressing the bare Green's function G_0 into G_1. Then Σ will be represented as a sum of diagrams where the "truncated" Green's functions G_1 and the bare vertices determined both by the terms I_1 and I_2 in the effective action figure. Note that the "complete" correlation function $\langle v_i v_k \rangle$ is not equal to zero, whereas the "complete" correlation function $\langle pp \rangle$ is zero.

Let us take G_1 in the form (14) with $z = \frac{d+2}{2}$ (3). Simple dimension estimations show that then any contribution to both Σ_1 and Σ has the form (13) with the same index z. From this one could conclude that the above assumption is self consistent and consequently (3) is the true exponent for the KPZ model. But such a conclusion is potentially very dangerous. The problem is that in the diagrammatic series for Σ_1 or Σ ultraviolet logarithmic divergences might appear which can change the exponent of the solution [31] or even destroy scaling behavior. The famous peculiarity of KPZ model is that these divergences do not appear in the diagrammatic series both for the "truncated" and the "complete" problems what we are going to demonstrate (the absence of the divergences in the "complete" KPZ problem on another language was proved in [19]).

As it is well known a loop of a diagram give rise to an integration over a wave vector $\mathbf{q}$ in the corresponding analytical expression. Therefore we should prove the absence of the ultraviolet divergences in integrals corresponding to all loops of diagrams representing contributions to Σ. The diagrams are constructed from lines representing G_1-functions which we will mark by arrows directed from p to $\tilde{v}$. Then a loop constructed from lines directed clockwise or anticlockwise give zero contribution because of causality properties of G-functions [18]. Therefore we may consider only loops with some "inputs" and "outputs" where two G-lines begin or end (a loop with one "input" and one "output" is depicted in the figure input being designated by "in" and output by "out").



The "outputs" are produced only by the term I_1 (10) and "inputs" are produced both by the term I_1 (10) and by the term I_2 (11). We see that in the "input" and "output" vertices produced by I_1 the derivative ∇ acts on a field external to the loop. The same assertion is valid also for the "input" vertices produced by I_2 since this term can be rewritten in the form

$$I_2 \to \int dt\, dr\, \lambda T^2 (\frac{3}{2} p_i^2 \nabla_k p'_k - p_i p_k \nabla_k p'_i)\,, \tag{17}$$

where $'$ designates the field external to the loop. The property enables us to prove that at large q the integral corresponding to the loop behaves as

$$q^{d+z} \cdot q^{n-2m} \cdot q^{-nz}\,, \tag{18}$$

where n is the number of vertices of the loop and m is the number of "inputs" or "outputs". It is not very difficult to check that the exponent here is negative for $n \geq 2$, $m \geq 1$ and z determined by (3) if $d < 2$. Therefore the integral converges at large q.

Thus we proved that at $1 < d < 2$ there are no any ultraviolet divergence in the diagrammatic series for Σ_1 or Σ where the bare vertices and the dressed functions G_1 figure. It implies that the scaling solution with the value (3) of the dynamic index does actually exist.

Now we may return to the original variable h. Using the relations (7) we find

$$\int dt\, d\mathbf{r} \exp(i\omega t - i\mathbf{k}\mathbf{r}) F(t, \mathbf{r}) = k^{-2-z} f(\omega/\mu_k)\,, \tag{19}$$

which is a consequence of (14,12).

For $d = 2$ numerical experiments show the value of z between 1.6 and 1.7 [13-15] whereas the solution (3) gives $z = 2$ at $d = 2$ not coinciding with above value. The reason of this contradiction is that the system of diagrammatic equations is very complicated and really a solution with another exponent or even with a nonscaling behavior may be realized, our assertion concerns only the existence of quasiequilibrium solution. Therefore the observed value of the index z can not be find in the framework of $1 + \epsilon$-expansion. The quasiequilibrium solution might be realized as a metastable solution at some conditions, for example at different character of short wavelength terms in the dynamic equation for h.

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