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CLUSTERS WITH MAGIC NUMBERS OF VORTICES IN ROTATING SUPERFLUIDS

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We discuss the internal structure of a cluster containing a large number of vortices in rotating superfluids. Clusters with magic numbers of vortices, in which the triangular vortex array in the interior of the cluster is continuously deformed into a system of concentric layers of vortices at the cluster boundary, are considered. The elasticity theory is applied for estimation of the cluster energy.

Recently experiments with finite vortex clusters in isotropic superfluid ^{3}He-B have been performed [1], which revived the interest to the problem of the structure of finite clusters. The same problem arises in mesoscopic systems with a finite number of costituent particles: nucleons, atoms, molecules, ions trapped in confining fields (Coulomb clusters, see Ref.[2] and references there). The latter system has much in common with the vortex cluster, since a free-standing vortex cluster with no contact to parallel walls in a rotating container is equivalent to a nonneutral two-dimensional classical plasma, which is a finite system of two-dimensional (negatively) charged particles confined in an infinite background of homogeneous (positive) charge.

Numerical calculations for vortices in rotating superfluids [3] and for Coulomb systems [2] have revealed that clustering occurs in different geometries and symmetries, but with features common for both systems. This includes "magic numbers" and well defined shell structure. In a cluster with large number N of vortices concentric circular shells are formed, while in Coulomb clusters the shells are spherical. In both cases the shape is dictated by the long-range Coulomb interaction of particles with each other and with the background charge.

In a vortex cluster with a magic number of vortices (see Fig.1 and Ref.[3]) the crystalline order of a triangular lattice exists only well inside the cluster. However, this crystalline order is extended in some way towards the edge of the cluster: in a "magic" cluster the number of vortices in the shells reproduces that of the triangular lattice: the first shell contains 1 vortex, the second 6, and the k'th shell contains $6(k-1)$ vortices. As a result a magic cluster - cluster with k compl etely closed shells - contains

$$N_k = 3k(k-1) + 1 \tag{1}$$

vortices.

An important property of the magic vortex clusters is related to the variation of an extra energy $\Delta f(N)$ as a function of N. $\Delta f(N)$ is defined as the energy of the cluster of N vortices with respect to the energy of N vortices in the equilibrium triangular lattice of an infinite vortex array. The formation of a

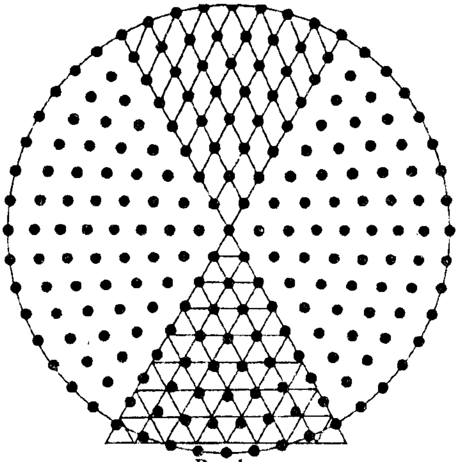

Рис.1

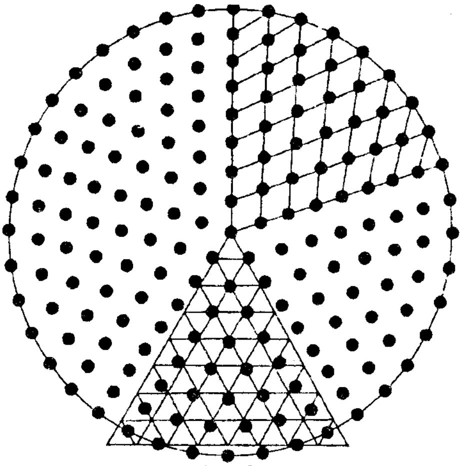

Рис.2

Fig.1. Magic cluster of 217 vortices numerically calculated by Campbell and Ziff [3]. Concentric rings on the boundary of the cluster are obtained by an elastic deformation of the triangular vortex lattice shown in the lower sector
Fig.2. Construction of the magic cluster of 181 vortices with 5-fold symmetry

same result for the energy excess over the energy of bcc lattice has been obtained in spherical Coulomb crystals [4]).

This can be explained in the following way: the magic clusters are obtained if an initial perfect monocrystal in the form of a hexahedron is continuously deformed by external superflow in such a way that its boundary becomes circular (Fig.1). In this scenario two main requirements (concentric circular shells on the boundary and triangular symmetry in the bulk) are fused together in a continuous way, i.e. without breaking or melting of the initial crystalline order. The "atomic" smoothness of crystal facets transforms to the "atomic" smoothness of the concentric circular layers on the boundary. The extra energy, which appears due to the finiteness of the cluster, is thus the elastic deformation energy, $\Delta f(N) = E_{el}$. As a result $\Delta f(N)$ is proportional not to the area of the cluster boundary ($\sim \sqrt{N}$), but to the total volume of the cluster ($\sim N$). Nevetherless, since the deformations are small, this scenario works even at large N.

In mesoscopic nuclei (with a large but finite number of nucleons), either the theory of continuous media, which properly takes into account the finite size of the system, or the shell model have been used. In the mesoscopic Coulomb clusters the shell model has been explored [2, 4, 5] and this model can be well applied to the vortex clusters. However, in the case of magic vortex (and magic Coulomb) clusters, where the deformation of initial lattice is elastic, the proper theory should be elasticity theory. The first attempt to apply the elasticity theory to the investigation of the cluster structure was made in Ref.[6].

Let us estimate $\Delta f(N)$ applying for simplicity the linear theory, which is justified because the displacements on the boundary are small compared with the radius R of the cluster, $u \leq 0.07R$, while large deformations occur only in the vicinity of the corners of the undeformed hexahedron. The elastic energy is

$$E_{el} = G\int d^2r(\vec{\nabla}\times \mathrm{u})^2 = G\int d^2r(\Delta\Psi)^2 \quad , \tag{2}$$

where $\mathrm{u}(r,\phi)$ is the distortion and $G=\rho_s\kappa\Omega/16\pi$ the shear modulus, the rigidity, which leads to Tkachenko waves [7]. As in Ref.[3] we use the energy unit $\rho_s\kappa^2/4\pi$, where the circulation quantum $\kappa = h/2m_3$ for ^{3}He and $\kappa = h/m_4$ for ^{4}He. We also take into account that the vortex array is incompressible, $\vec{\nabla}\cdot\mathrm{u}=0$, and introduce the potential Ψ, such that $\mathrm{u}=\hat{z}\times\vec{\nabla}\Psi$. The displacement $\mathrm{u}(r,\phi)$ satisfies the boundary conditions corresponding to the transformation of the facets of the hexahedron into a circular shell. In the linear approximation the boundary conditions at $r=R$ are:

$$\vec{u}(R,\phi) = \hat{r}\ \ R(1-\frac{\sqrt{\pi/(2\sqrt{3})}}{\cos\phi}) \quad . \tag{3}$$

The general solution of the equation $\Delta^2\Psi=0$ for Ψ is:

$$\Psi(r,\phi) = \sum_n [A_n(\frac{r}{R})^{6n} + B_n(\frac{r}{R})^{6n+2}]\sin 6n\phi \quad , \tag{4}$$

and from the boundary conditions in Eq. (3)

$$-\partial_\phi\Psi(R,\phi) = -R^2\sum_n 6n[A_n+B_n]\cos 6n\phi = R^2(1-\frac{\sqrt{\pi/(2\sqrt{3})}}{\cos\phi}) \quad ,$$

$$R\partial_r\Psi(r,\phi)|_{r=R} = R^2\sum_n[6nA_n+(6n+2)B_n]\sin 6n\phi = 0 \quad , \tag{5}$$

one finds

$$B_n = \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}(1-\frac{\sqrt{\pi/(2\sqrt{3})}}{\cos(\phi/6)})\cos n\phi \quad . \tag{6}$$

Since

$$\Delta\Psi(r,\phi) = 4R^2\sum_n(6n+1)B_n(\frac{r}{R})^{6n}\sin 6n\phi \quad , \tag{7}$$

the elastic energy is

$$E_{el} = N\sum_n(6n+1)B_n^2 = N\frac{\pi}{2\sqrt{3}}\sum_1^\infty(6n+1)[\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\frac{\cos n\phi}{\cos(\phi/6)}]^2 \quad , \tag{8}$$

where we have taken into account that in dimensionless notation $4\pi R^2G=N$. The final result for $\Delta f(N)=E_{el}$ in the linear approximation is $0.0073N$.

This is in a reasonable agreement with numerical results [3], obtained for magic clusters of 61, 91, 127, 169 and 217 vortices. For the vortex patterns without inelastic deformations, $\Delta f(N)/N$ converges to 0.0059, while for vortex patterns with plastic deformation - the outmost ring is rotated by one half of the intervortex distance - $\Delta f(N)/N$ is slightly less. The small discrepancy arises because nonlinear corrections are important especially near the corners of the hexagon. The linear theory should work better in a 3-dimensional system, because one can choose polyhedrons which are very close to the spherical shape.

For clusters far from the magic ones, other geometries and therefore different scenarios can be important. The simplest alternative is to cut a circular cylinder from the perfect monocrystal and then melt the region close to the boundary to transform it into a system of circular rings. This should lead to an extra energy $\Delta f(N)$ proportional only to the circumference of the cluster, i.e. $\propto \sqrt{N}$. (It was argued in Refs.[6] and [7] on the basis of elasticity theory that an "atomic" rearrangement of vortices in the surface region can be also accompanied by elastic deformations in the bulk and can lead to $\Delta f(N) \propto N$.) Nevertheless, for not very large clusters and for N close to the magic number the elastic deformation scenario appears to be more profitable, because the transformation of a hexahedron into circular shells requires rather small elastic energy and therefore the bulk energy related to the elastic deformation can be smaller than the surface energy related to the "atomic" rearrangement.

Other geometrical constructions are possible with 5-fold and 7-fold symmetries prohibited for periodic crystals, they correspond to icosahedral symmetry in 3-dimensions. The cluster consists of 5 (Fig. 2) or 7 (Fig. 3) sectors, with each sector (shown in the right half of the cluster) being obtained by elastic deformation from the rectilinear triangle of a regular vortex array shown in the lower part. Two neighbouring sectors in this construction are separated by a twin boundary. The magic numbers for these clusters are $1 + (5/2)k(k-1)$ and $1 + (7/2)k(k-1)$ correspondingly.

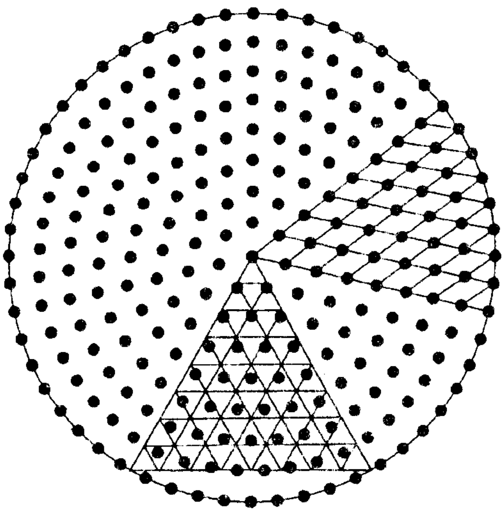

Fig.3. Construction of the magic cluster of 253 vortices with 7-fold symmetry

The magic clusters can generate other families of clusters, obtained from the magic cluster by introducing vacancies, interstitials, dislocations and other defects. While vacancies and intrusions increase $\Delta f(N)/N$, a dislocation introduced into a strained system can in principle decrease the energy. Two families of clusters can be obtained by introducing a dislocation into the center of the vortex pattern. Since the dislocation corresponds to the insertion or deletion of a row, these familie s have the following numbers of vortices: $N_k^+ = N_k + (k-1)$ for inserted and $N_k^- = N_k + k$ for deleted rows. From the example with $k = 3$ one can see that the dislocation can really decrease the energy: $\Delta f(N_3^+ = 21) = 0.191 < \Delta f(N_3 = 19) = 0.193$ (see Ref.[3]).

Since the clusters with magic vortex numbers correspond to local minima of the energy as a function of N we hope that they can be observed in superfluid ^{3}He-B where the sensitivity of the NMR signal allows to resolve a single event of vortex nucleation [8]. The elasticity theory can be also applied to Coulomb clusters, in particular to the classical nonneutral plasma confined in magnetic field, for which the shell model in terms of cylindrical shells has been discussed in Ref.[5].

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