Pis'ma v ZhETF, vol. 77, iss. 6, pp. 376 – 379

On integration of some classes of $(n+1)$-dimensional nonlinear Equations of mathematical physics

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Submitted 6 February 2003

This paper represents method for construction of the families of particular solutions to some new classes of $(n+1)$-dimensional nonlinear Partial Differential Equations (PDE). Method is based on the specific link between algebraic matrix equations and PDE. Admittable solutions depend on arbitrary functions of n variables. Examples of deformed Burgers type equations are given.

PACS: 05.45.–a

1. Introduction.

Many different methods have been developed for analytical investigation of nonlinear Partial Differential Equations (PDE) during last decades. Especially attractive are methods for study of so-called completely integrable systems. The particular interest to these equations is enhanced due to their wide range of application in physics. We emphasize different dressing methods, which are based on fundamental properties of linear operators, either differential or integral: Zakharov-Shabat dressing method [1, 2], $\bar\partial$-problem [3 – 6], Sato theory [7].

We suggest the method for construction of the families of particular solutions to some new classes of $(n+1)$ dimensional nonlinear PDE, $n \geq 2$. It is based on general properties of linear *algebraic* matrix equations [7 – 9]. In general, this method supplies the family of solutions depending on the set of arbitrary functions of n variables for $(n+1)$-dimensional PDE. The represented method also can be applied to the classical $(2+1)$-dimensional PDE integrable by the Inverse Scattering Technique (IST). In this case our algorithm is similar to the algorithm represented in refs.[7, 10].

We discuss general algorithm relating linear algebraic equations with nonlinear PDE. Then we show that these PDE are compatibility conditions for appropriate overdetermined linear system of equations having different structure in comparison with classical isospectral problem. Some examples of (2+1)-dimensional equations are represented.

2. General algorithm.

As mentioned above, our algorithm is based on the fundamental properties of linear matrix algebraic equation

$$\Psi U = \Phi, \qquad (1)$$

where $\Psi = \{\psi_{ij}\}$ is $N \times N$ nondegenerate matrix, U and Φ are $N \times M$ matrices, $M < N$. Namely, the solution of this equation is unique, and hence, the homogeneous equation with matrix Ψ has only the trivial solution. Thus, if we find transformation which maps the nonhomogeneous Eq.(1) into the homogeneous equation $\Psi \tilde U(U) = 0$, then $\tilde U(U) = 0$.

It may be shown that such transformations can be performed using differential operators having special structure. For this purpose let us introduce two types of additional parameters $x = (x_1, \dots, x_n)$ $(n = \dim(x))$ and $t = (t_1, t_2, \dots)$ by the following systems:

$$\Psi_{x_i} = \Psi B_i + \Phi C_i, \;\; i = 1, \dots, n \qquad (2)$$

(B_i and C_i are constant $N \times N$ and $M \times N$ matrices respectively) and

$$\mathcal{M}_i \Psi = 0, \quad \mathcal{M}_i \Phi = 0, \;\; \mathcal{M}_i = \partial_{t_i} + L_i, \qquad (3)$$

where L_i are arbitrary linear differential operators having derivatives with respect to variables x_j and constant scalar coefficients, so that the system (2) is compatible with the system (3). For the sake of simplicity in this paper we use only one parameter t, omit subscripts in the Eq.(3) and use n-dimensional Laplacian for L: $\mathcal{M} = \partial_t + \sum_{k=1}^n \alpha_k \partial^2_{x_k}$. Hereafter indexes i, j and k run values from 1 to n unless otherwise specified.

Let us study compatibility conditions for the system (2) itself, which has the following form:

$$\begin{aligned}&(\Psi B_j + \Phi C_j))B_i + \Phi_{x_j} C_i = \\ &= (\Psi B_i + \Phi C_i))B_j + \Phi_{x_i} C_j.\end{aligned} \qquad (4)$$

Require that matrices B_i and C_i satisfy two conditions

$$C_j B_i - C_i B_j = 0, \;\; B_j B_i - B_i B_j = 0, \qquad (5)$$

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and matrices C_j have the following structure: $C_j = \left[P_j \mid 0_{M,N-R} \right]$, $R \le M$, where P_j are $M \times R$ matrices with rang R and $0_{A,B}$ means $A \times B$ zero matrix. Then Eq.(4) is simplified to the next one:

$$\Phi_{x_i} P_j - \Phi_{x_j} P_i = 0, \qquad (6)$$

which results in the first nonlinear equation for U owing to the Eqs.(1) and (2):

$$(B_j + UC_j)UP_i + U_{x_j}P_i = (B_i + UC_i)UP_j + U_{x_i}P_j. \quad (7)$$

Let us show that another nonlinear matrix equation can be derived using operator $\mathcal{M}$. For this purpose we apply operator $\mathcal{M}$ to both sides of the Eq.(1) and use Eqs.(2) and (3):

$$0 = \mathcal{M}\Phi = (\mathcal{M}\Psi)U + \Psi\mathcal{M}U + 2\sum_{k=1}^n \alpha_k \Psi_{x_k} U_{x_k} = \\ = \Psi\left(\mathcal{M}U + 2\sum_{k=1}^n \alpha_k (B_k + UC_k)U_{x_k}\right) = 0.$$

Since $\det(\Psi) \neq 0$, one has the second nonlinear equation for the matrix U:

$$U_t + \sum_{k=1}^n \alpha_k \left(U_{x_k x_k} + 2(B_k + UC_k)U_{x_k}\right) = 0. \qquad (8)$$

Combining Eqs. (7), (8) and using condition (5) one can eliminate matrices B_i and receive system of equations for matrix V composed by the first R rows of the matrix U. First equation exists for any n. To derive it let us eliminate operators B_i from the Eq. (7) using Eqs. (5) and (8):

$$P_i\left(U_{j_t} + \sum_{k=1}^n \alpha_k \left(U_{j_{x_k x_k}} + 2U_k U_{j_{x_k}}\right)\right) - \\ - P_j\left(U_{i_t} + \sum_{k=1}^n \alpha_k \left(U_{i_{x_k x_k}} + 2U_k U_{i_{x_k}}\right)\right) + \\ + 2\sum_{k=1}^n \alpha_k P_k \left(U_{i_{x_j}} - U_{j_{x_i}} + [U_j, U_i]\right)_{x_k} = 0. \qquad (9)$$

Another equation exists if only $n > 2$. In this case we can derive the matrix equation without operators B_i using any three Eqs. (7) with pairs of indexes (i,j), (j,k), (k,i) and relations (5):

$$\sum_{perm} P_i\left(U_{j_{x_k}} - U_{k_{x_j}} + [U_k, U_j]\right) = 0, \;\; U_i = VP_i, \quad (10)$$

where sum is over cycle permutation of indexes i, j and k. In general, both Eqs.(9) and (10) should be considered simultaneously. Otherwise the nonlinear system may not be completed.

Thus Eqs.(9) and (10) do not depend on the parameter N (which characterizes dimensions of the matrices in the Eq.(1)) as well as on the matrices B_i, i.e. N is arbitrary positive integer, B_i are $N \times N$ matrices fitting relations (5).

Arbitrary functions of variables x_j ($j = 1, \dots, n$) appear in the solution V due to the matrix function Φ, defined by the system (6). The number of arguments in the arbitrary functions as well as the number of these functions is defined by the particular choice of the matrices P_j and dimension n of x-space. For n-dimensional x-space and $R < M$ we are able to represent examples with N arbitrary functions of n variables. But in particular cases the situation may be different (see **Examples**). For instance, if $R = M$ then Φ may depend at most on $N \times M$ arbitrary scalar functions of *single* variable, which is in accordance with [7]. Detailed discussion of this problem is left beyond the scope of this paper.

2.1. On the operator representation of PDE. Classical nonlinear (2+1)-dimensional systems integrable by the IST are inside of the family of equations introduced in the previous section. To clarify this statement we derive the overdetermined linear system of PDE with compatibility condition in the form of Eqs.(9) and (10) and compare it with classical isospectral problem. First, introduce arbitrary $R \times N$ matrix function $\mathbf{R}(\lambda)$ of the additional complex parameter λ. Multiply Eqs.(7) and (8) by $\mathbf{R}(\lambda)\exp\eta$, $\eta = \sum_{k=1}^n B_k x_k - \left(\sum_{k=1}^n \alpha_k B_k^2\right) t$, from the left and introduce function $\hat{\Psi} = \mathbf{R}e^{\eta}U$. We get after transformations:

$$\hat{\Psi}_{x_j} P_i - \hat{\Psi}_{x_i} P_j = \hat{\Psi} P_i U_j - \hat{\Psi} P_j U_i, \qquad (11)$$

$$\hat{\Psi}_t + \sum_{k=1}^n \alpha_k \left(\hat{\Psi}_{x_k x_k} + 2\hat{\Psi} P_k V_{x_k}\right) = 0. \qquad (12)$$

If $R = M$, t.e. all P_j are square nondegenerate matrices, then the system (11), (12) is equivalent to the classical $M \times M$ overdetermined linear system for correspondent (2+1)-dimensional integrable system. In fact, in this case $\det P_j \neq 0$, so that one can express all derivatives of $\hat{\Psi}$ with respect to x_j, $j > 1$ through the derivatives of $\hat{\Psi}$ with respect to x_1 using Eq.(11). Both Eqs.(11) and (12) are $M \times M$ matrix Eqs. for $M \times M$ matrix function $\hat{\Psi}$. Thus Eq. (11) can be taken for the spectral problem while Eq. (12) represents evolution part of the overdetermined linear system. For instance, if $M = R = n = 2$, $\alpha_1 = 1$, $\alpha_2 = 0$, and P_i have the form (14), then the compatibility condition of the linear system (11), (12) is given by the Eqs.(15).

The situation is quite different if $R < M$. $\hat{\Psi}$ is $R \times M$ matrix function, while (11) is $R \times R$ matrix Equation.

Thus it can not be taken for the spectral problem. Moreover, Eq.(12) involves all derivatives appearing in the operator M. So, we have $(n+1)$ dimensional Equations. Important fact is that solution of these equations depend on arbitrary functions of n variables, which has been shown in the end of the previous section. Below we represent example of $(2+1)$-dimensional system of this type.

2.2. Examples.

Let $n=2$, $\alpha_1=1$, $\alpha_2=0$. Then Eq.(9) reduces in

$$P_1\left(U_{2t}+2U_{1x_1x_2}-U_{2x_1x_1}+2(U_2U_1)_{x_1}-2U_{1x_1}U_2\right)-\\ -P_2\left(U_{1t}+U_{1x_1x_1}+2U_1U_{1x_1}\right)=0, \qquad (13)$$

while Eq.(10) does not exists. First we point on two trivial reductions of the Eq.(13).

1. Let matrix A exist such that $AP_1=0$ and $AP_2=I_R$ (I_R is $R\times R$ identity matrix). Multiplying Eq.(13) by A from the left one receives matrix Burgers equation for U_1.

2. If $R=M=2$ and

$$P_1=I_2,\ \ P_2=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ \ V=\begin{bmatrix}v_{11}&v_{12}\\v_{21}&v_{22}\end{bmatrix}, \qquad (14)$$

then Eq.(13) is equivalent to the next system:

$$\begin{aligned}&r_t-r_{x_1x_2}-2rw_{x_1x_2}=0,\\&q_t+q_{x_1x_2}+2qw_{x_1x_2}=0,\\&w_{x_1x_1}-w_{x_2x_2}=qr,\end{aligned} \qquad (15)$$

where functions r and q are related with elements of the matrix V by the formulae $v_{11}=\frac{1}{4}(r-q+2w_{x_1})$, $v_{21}=\frac{1}{4}(r+q+2w_{x_2})$, $v_{12}=\frac{1}{4}(-r-q+2w_{x_2})$, $v_{22}==\frac{1}{4}(-r+q+2w_{x_1})$. Eq.(15) becomes Devi-Stewartson Equation after reduction $r=\psi$, $q=\bar\psi$, $t_1=it$, where $i^2=-1$, bar means complex conjugated value.

Essentially different situation is described in the next example.

3. Let $M=3$, $R=2$, $P_1=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$, $P_2=$ $=\begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}$. It is possible to construct solutions to the system (13) depending on N arbitrary functions of two variables. In fact, let $N=3m+2$, $m=1,2,\ldots$, matrices B_i be defined, for instance, by the following system:

$$B_j=\begin{bmatrix}0_{2,2}&b_{j1}&0_{2,N-5}\\0_{3(m-1),2}&0_{3(m-1),3}&b_{j2}\\0_{3,2}&0_{3,3}&0_{3,N-5}\end{bmatrix},$$

$$b_{11}=\begin{bmatrix}1&1&1\\0&0&0\end{bmatrix},\quad b_{21}=\begin{bmatrix}0&0&0\\1&1&1\end{bmatrix},$$

$$b_{j2}=\mathrm{diag}(\underbrace{A_j,A_j,\ldots}_{m-1}),\ \ j=1,2,$$

$$A_1=\begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix},\ A_2=\begin{bmatrix}1&-1&0\\0&1&-1\\-1&0&1\end{bmatrix}.$$

$\Phi=[\phi_1\quad\phi_2\quad\phi_3]$, where ϕ_k are N-dimensional columns. Then Eq.(6) can be written in the form $\phi_{1x_2}=\phi_{2x_1}$, $\phi_{2x_2}=\phi_{3x_1}$, i.e. $\phi_1=S_{x_1x_1}$, $\phi_2=S_{x_1x_2}$, $\phi_3=S_{x_2x_2}$, where S is arbitrary function of variables x_1 and x_2. In view of Eqs.(3), we can write for Φ: $\Phi=\int_{-\infty}^{\infty}c(k_1,k_2)[k_1^2\quad k_2k_1\quad k_2^2]\exp[k_1x_1+k_2x_2-k_1^2t]dk_1dk_2$, where $c(k_1,k_2)$ is arbitrary column of N elements. Function Ψ (solution of the system (2) and (3)) can be represented in the form $\Psi=(\Psi_0+\tilde\Psi)e^{\eta}$, $\eta=B_1x_1+B_2x_2-B_1^2t$, where Ψ_0 is arbitrary constant $N\times N$ matrix and $\tilde\Psi$ is taken in the form $\tilde\Psi=\partial_{x_1}^{-1}\Phi C_1e^{-\eta}$. Then $U=\Psi^{-1}\Phi$ depends on arbitrary $N\times 1$ matrix function $c(k_1,k_2)$ of two variables.

The appropriate nonlinear system (13) has rather complicated form. To simplify it for application purpose we suggest two methods.

1. *Multi-scale expansion.* Introduce parameter $\epsilon\ll 1$ and new scales for coordinates: $\partial_{x_j}\to\epsilon\partial_{x_j}$ and $\partial_t\to\to\epsilon^2\partial_t$. Let us consider matrix V in the following form

$$V=\begin{bmatrix}2+\epsilon^2u_1&\epsilon^2v_1&\epsilon+\epsilon^2w_1\\\epsilon u&1+\epsilon v&\epsilon^2w_2\end{bmatrix}.$$

Then we can write the Eq.(13) in the form of two-component Burgers equation with perturbation up to the order $O(\epsilon^2)$

$$\begin{aligned}2u_{x_1}&=-\epsilon(u_t+2vu_{x_1}+u_{x_1x_1})+\\&+\epsilon^2(4uvv_{x_1}-2uu_{x_1x_2}+2uv_{x_1x_1}),\\2v_{x_1}&=-\epsilon(v_t+2vv_{x_1}+v_{x_1x_1})+2\epsilon^2uu_{x_1}.\end{aligned}$$

One can eliminate derivatives with respect to x_1 from the right hand sides (terms with ϵ^3 appear): $2u_{x_1}==-\epsilon u_t+\epsilon^2vu_t-\epsilon^3\left(v^2u_t+2uvv_t+1/4u_{tt}-uu_{x_2t}\right)$, $2v_{x_1}=-\epsilon v_t+\epsilon^2vv_t-\epsilon^3\left(uu_t+v^2v_t+1/4v_{tt}\right)$.

2. *Reductions.* There is a wide class of reductions in the form of additional differential Equations for the functions ϕ_i, $i=1,2,3$, compatible with two equations written above. They reduce the number of independent functions in the Eq.(13), although freedom

in construction of solutions is reduced as well. For instance, let $\phi_2 = \phi_{1x_1}$ and $S_{x_2x_1} = S_{x_1x_1x_1}$. One has additional equation for the column of the matrix $V = [V_1\ V_2\ V_3]$: $V_2 = (B_1 + U_1)V_1 + V_{1x_1}$. Using first column of the Eq.(8) (with $n = 2$, $\alpha_1 = 1$, $\alpha_2 = 0$) one can eliminate B_1 from the above equation to result in: $V_{1t} + V_{1x_1x_1} + 2U_1V_{1x_1} = -2V_{2x_1} + 2(U_1V_1)_{x_1} + 2V_{1x_1x_1}$. With this equation the nonlinear system (13) is reduced to:

$$q_t + q_{x_1x_1} + 2q\varphi_{x_1x_1} = 0, \quad \varphi_t + \varphi_{x_2} = 0,$$

$$\varphi_{x_2x_2} + 2\varphi^2_{x_1x_1} + \Big[\varphi_{x_1x_1x_1} - 2\varphi_{x_1x_2} -$$

$$- 2\varphi_{x_2}\varphi_{x_1} + 2\varphi_{x_1}\varphi_{x_1x_1} + \frac{2}{3}\varphi^3_{x_1}\Big]_{x_1} = 0,$$

where φ and q are expressed through the first column elements of V: $\varphi_{x_1} = v_{11}/(1 - v_{21})$, $q = v_{21}/(1 - v_{21})$. Note, that the last equation has no Lagrangian. It may be given the form of nonlinear Burgers equation with nonlocal correction after applying operator $\partial^{-2}_{x_1}$.

The represented version of the dressing method works for wide class of $(n + 1)$-dimensional PDE. It supplies solutions depending on arbitrary functions of n variables provided $R < M$. In the case $R = M$ solution depends on arbitrary functions of single variable. By construction, equations have infinite number of commuting flows introduced by the Eqs.(3). They also are compatibility conditions for some specific linear overdetermined systems, which are equivalent to the classical isospectral problems if only $R = M$.

The work is supported by RFBR grants #02-01-06059 and #00-15-96007. Author thanks Prof. S. V. Manakov and Dr. Marikhin for discussions and Dr. Bogdanov for useful comments.

1. V. E. Zakharov and A. B. Shabat, Funct. Anal. Appl. **8**, 43 (1974).
2. V. E. Zakharov and A. B. Shabat, Funct. Anal. Appl. **13**, 13 (1979).
3. V. E. Zakharov and S. V. Manakov, Funct. Anal. Appl. **19**, 11 (1985).
4. L. V. Bogdanov and S. V. Manakov, J. Phys. A: Math. Gen. **21**, L537 (1988).
5. B. Konopelchenko, *Solitons in Multidimensions*, World Scientific, Singapore, 1993.
6. A. I. Zenchuk, J. Math. Phys. **41**, 6248 (2000).
7. Y. Ohta, J. Satsuma, D. Takahashi, and T. Tokihiro, Progr. Theor. Phys. Suppl. **94**, 210 (1988).
8. A. I. Zenchuk, arXiv:nlin.SI/0202053 v1 (2002).
9. A. I. Zenchuk, arXiv:nlin.SI/0210031 v1 (2002).
10. F. Guil, M.Mañas, and G.Álvarez, Phys. Lett. **A190**, 49 (1994).