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Phase transition in a self-repairing random network

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We consider a network, bonds of which are being sequentially removed; that is done at random, but conditioned on the system remaining connected (Self-Repairing Bond Percolation SRBP). This model is the simplest representative of a class of random systems for which forming of isolated clusters is forbidden. It qualitatively describes the process of fabrication of artificial porous materials and degradation of strained polymers. We find a phase transition at a finite concentration of bonds $p = p_c$, at which the backbone of the system vanishes; for all $p < p_c$ the network is a dense fractal.

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Properties of networks, bonds (or nodes) of which are being removed randomly, have been intensively studied during past 50 years. In the standard formulation of the problem the bonds (or sites) are removed totally at random, and the percolation transition takes place at a certain concentration p_c of remaining bonds (or concentration x_c of remaining sites). For $p > p_c$ there exists an "infinite" cluster that contains a finite fraction of all bonds in the system and spans homogeneously through the entire network. At $p < p_c$ the infinite cluster does not exist and only finite ones are present. The percolation phase transition and corresponding critical phenomena are well studied (see, e.g., [1, 2]).

However, there are many physical systems which cannot be described by the standard percolation theory. In particular, there are important cases when finite clusters can not appear at all. As an example, consider a technologically important process of pore forming (i.e., fabrication of a porous material; see, e.g., [3]). It can be viewed as gradual removal of grains of a pore-former (carbon, which can be burned out, or a soluble polymer) from a mixture of the pore-former with grains of a matrix material (a metal). Due to mechanical instability of finite clusters, they immediately fall down onto the surrounding matrix and stick to it. Thus at any stage of the process the remaining grains form a single "infinite cluster" and the percolation transition is impossible. The properties of this single cluster are nontrivial: it appears that there is a topological phase transition at a finite concentration of remaining grains x_c, below which the system becomes "cracked" and its mechanical and conducting properties degrade catastrophically.

In the present paper we introduce a simple model which possesses the main property of the above systems (existence of only one cluster) and allows rigorous analysis. The model is very similar to the standard bond percolation: starting from a full lattice, at each step one of the remaining bonds is randomly chosen for removal. But after its removal the system is checked for the existence of finite clusters: if such ones are present, then the removed bond is restored (i.e., the last removal is cancelled) and the process goes on to the next step. It seems natural to call this model a "self-repairing bond percolation" (SRBP).

Apart from being relevant to pore-forming, the SRBP model may also be viewed as a model for polymer degradation (see, e.g., [4]). Consider a random network consisting of irregularly cross-linked polymer chains. Suppose that this system is subjected to random external perturbation (e.g., UV-radiation) that can destroy the cross-links. The radiation damage may be repairable: attraction between individual chains tends to reestablish the damaged link. However, sometimes that appears to be impossible, since internal strains in the chains may drive the two chains apart as soon as the link between them is damaged. Thus it seems reasonable to assume that all strained links are vulnerable to radiational damage, while unstrained ones are "immune" to it. Of course, finding out which links in a random network are strained and which are not is a formidable task. But in any case the links which are the only bridges connecting otherwise isolated clusters are never strained. These links will be repaired after possible removal, in accordance with the definition of the SRBP model.

Our model is also relevant to river network formation. In two dimensions it is in fact dual to the "loopless percolation" model that was studied numerically in [5].

Let the fraction of the remaining bonds be p. We will be interested in average properties of the system as function of p. In particular, we will study the conductivity $\sigma(p)$ and "the minimal chemical path" $\ell(R, p)$, the latter

being an ensemble-averaged length of the shortest path going via bonds and connecting two points separated by euclidean distance R.

The first obvious observation about the SRBP model is that there exists some minimal possible $p = p_{\rm tree} \equiv 2/z$ (z being the coordination number of the lattice) at which the process of bond removal stops: for a connected graph one necessarily has $p \geq p_{\rm tree}$. At $p = p_{\rm tree}$ the remaining bonds constitute a spanning tree (ST), a connected graph with no cycles and all lattice sites as vertices. As it is well known, the probability of generating a given ST at the end of our process is related to the Minimal Spanning Tree (MST) problem (see, e.g., [6]). In particular, the minimal chemical path is fractal: $\ell(R, p = p_{\rm tree}) \propto R^{d_{\rm min}^{\rm MST}}$ with $d_{\rm min}^{\rm MST} > 1$. Obviously, for a tree one has $\sigma(p = p_{\rm tree}) = 0$.

We will show that actually the minimal chemical path is fractal and the specific conductivity of the system is zero not only at $p = p_{\rm tree}$, but also within a finite interval $p_{\rm tree} \leq p \leq p_c$. The corresponding phase we will call the "tree-like phase", in contrast to the "solid phase" existing at $p_c \leq p < 1$.

To prove the above statement, we use a mapping to the standard percolation. Suppose that initially all the bonds of the system are black. If at some step a removal of a certain bond must be cancelled, we restore the bond but change its color to grey. Then for any fraction p of remaining bonds we have fractions $b = b(p)$ of black and $g = p - b(p)$ of grey bonds remaining. It is easy to show that $b(p)$ is a monotonically increasing function with the following asymptotics:

$$b(p) \approx \begin{cases} p & \text{for } 1 - p \ll 1, \\ 0 & \text{for } p \to p_{\rm tree}. \end{cases} \qquad (1)$$

Clearly, a grey bond may never be removed, and at $p = p_{\rm tree}$ all bonds are either removed or grey. It is easy to see that the backbone of the entire (black and grey) network coincides with the backbone of the black subsystem. Indeed, no grey bond can belong to backbone: since its removal produces an isolated finite cluster, such a bond belongs to a dangling end. Note that black bonds are removed *totally at random*, hence the behavior of the black subsystem is identical to that of the standard bond-percolation system. In particular, the backbone vanishes at the percolation point, where $b = p_{\rm perc}$. It follows that there exists a critical concentration of bonds p_c such that for $p < p_c$ the remaining bonds all belong to one infinite cluster (which has finite density), while this cluster *has no backbone*. The critical concentration is determined by the condition

$$b(p_c) = p_{\rm perc}, \qquad (2)$$

where $p_{\rm perc}$ is the percolation threshold for the standard bond percolation problem on the same lattice. On the other hand, the number of grey bonds in the system is equal to the number of finite black clusters: $g = n_{\rm cl}$ (see Fig.1). Thus the critical concentration p_c can be

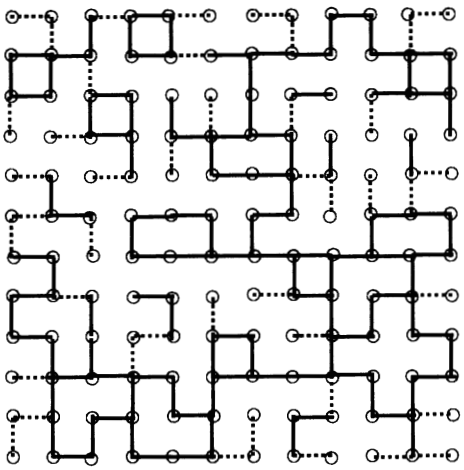

Fig.1. A typical configuration of black ang grey bonds at the critical concentration for 2D square lattice.

expressed solely through the characteristics of the standard percolation problem:

$$p_c = p_{\rm perc} + n_{\rm cl}^*, \qquad (3)$$

where $n_{\rm cl}^*$ is the number of finite clusters (per one bond of the initial lattice) at the critical point. The latter is known for many lattices; in particular, for the square lattice $n_{\rm cl}^* = (3\sqrt{3} - 5)/4$ (see [7, 8]) and $p_{\rm perc} = 1/2$, so

$$p_c = \frac{3\sqrt{3} - 3}{4} \approx 0.54904. \qquad (4)$$

In scaling theory of percolation the relations between different critical exponents are normally derived from considerations involving distribution function of finite clusters (see, e.g., [2]). In our model finite clusters do not exist whatsoever, but fortunately one can introduce blocks–alternative objects, which to some extent play the role of finite clusters and make it possible to develop the scaling theory. By definition, a *block* is a maximal subgraph that cannot be disconnected by deletion of a single vertex [9]. It is not difficult to show that either a block consists of a single bond (and its two ends), or any two bonds belonging to a block lie on a common cycle. Two distinct blocks may have at most one point in common; such a point is called an *articulation point*, and its deletion necessarily disconnects the system. Given a network, one can form a graph with blocks and articulation

points of the network as vertices, with two vertices connected if they correspond to an articulation point and a block that contains it. Such a *block graph* is always a tree; an example is shown in Fig.2.

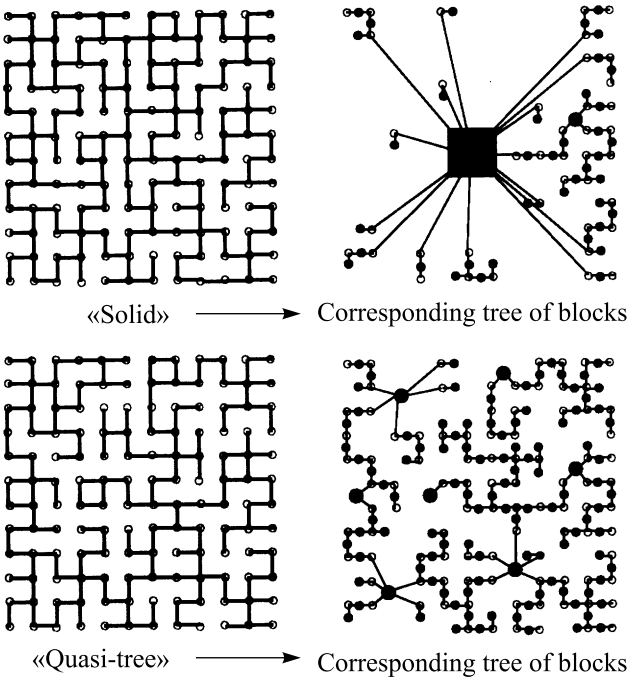

Fig.2. Typical configurations of bonds in the solid and tree-like phases (we do not distinguish black and grey bonds here), together with corresponding block graphs. Black square–the infinite block (the backbone); large solid circles–finite nontrivial blocks; small solid circles–trivial blocks; small open circles–articulation points

The backbone that exists in the solid phase constitutes the only infinite block in the system; it has finite density and contains infinite cycles. The backbone is linked to an infinite number of branches–dangling ends, each dangling end being a finite tree of finite blocks.

In the tree-like phase the infinite block collapses, so that there are only finite cycles in this phase. The corresponding bond configurations we will call quasitrees.

In the solid phase, the backbone becomes more loose as p approaches p_c. The fraction of bonds belonging to it tends to zero:

$$P_B(p) \propto (p - p_c)^{\beta_B}, \tag{5}$$

where β_B is the index of the backbone density for the standard percolation (in particular, $\beta_B \approx 0.48$ for $d = 2$, see, e.g., [2] and [10]).

The total number of dangling ends decreases as $p \to p_c$ from above, while the number of blocks in each dangling end and the number of bonds in a typical block increase and diverge as $p \to p_c$. It is convenient to introduce the distribution function of finite blocks consisting of l bonds:

$$n_l(p) \sim l^{-\tau} f\left[l(p - p_c)^{1/\sigma}\right], \tag{6}$$

where $f(x)$ is a universal function that decays exponentially at $x \gg 1$. Application of standard scaling arguments [2] to blocks leads to the following relations between the critical exponents:

$$\nu = \frac{\tau - 1}{\sigma d}, \qquad \xi(p) \sim (p - p_c)^{-\nu} \tag{7}$$

The length $\xi(p)$ characterizes correlations within the backbone, and since the backbone for the SRBP model is the same as for standard percolation, we conclude that the exponent ν for the SRBP model coincides with that for standard percolation.

The exponent γ that characterizes the behavior of the mean size $S(p)$ of finite blocks near p_c is

$$\gamma = \frac{3 - \tau}{\sigma}, \qquad S(p) \equiv \frac{\sum_l l^2 n_l(p)}{p - P_B(p)} \propto (p - p_c)^{-\gamma}. \tag{8}$$

Finally,

$$\beta_B = \frac{\tau - 2}{\sigma}, \qquad d_B = \frac{d}{\tau - 1}, \tag{9}$$

where d_B is the fractal dimension of the backbone at $p = p_c$.

Since the conduction process involves only the backbone, the conductivity of the SRBP model is identical to that of the standard percolation,

$$\sigma_{\mathrm{SRBP}}(p) \equiv \sigma_{\mathrm{perc}}[b(p)] \propto (p - p_c)^{\mu}, \tag{10}$$

hence the critical exponent μ is the standard one. For the minimal path length in the solid phase one has

$$\begin{gathered} \ell(R) = \frac{R}{v(p)}, \\ v(p) \sim \xi(p)^{1 - d_{\min}^{\mathrm{perc}}} \sim (p - p_c)^{-\nu + \nu d_{\min}^{\mathrm{perc}}}, \end{gathered} \tag{11}$$

where $d_{\min}^{\mathrm{perc}}$ is the graph dimension for the infinite cluster at the critical point for the standard percolation problem. As usual, the formula (11) is valid only for $R \gg \xi(p)$; in the opposite case $R \ll \xi(p)$ it should be substituted by the critical law

$$\ell(R) \propto R^{d_{\min}^{\mathrm{perc}}}. \tag{12}$$

For $R \sim \xi(p)$ the expressions (11) and (12) match.

At $p = p_{\mathrm{tree}}$ our system is reduced to the MST ensemble. In high dimensions $d > d_c$ a minimal spanning

tree on an *infinite* lattice may in fact have many components. It is believed (see [11]) that $d_c = 8$, and in $d < 8$ dimensions for almost all trees of the MST ensemble there exists a unique finite path $\mathcal{P}(A, B)$ connecting any two given sites A and B. This path is a certain non self-intersecting random walk with fractal dimension $d_f = d_{\min}$. While this dimension is known exactly for the 2D uniform spanning tree ensemble, only numerical estimates are available for the MST case: $d_{\min} \approx 1.22$ for $d = 2$, $d_{\min} \approx 1.42$ for $d = 3$, and $d_{\min} \approx 1.59$ for $d = 4$ (see [12]). Below we demonstrate that the graph dimension is the same *throughout the entire tree-like phase*:

$$d_{\min}(p) = d_{\min}^{\mathrm{MST}}, \qquad \text{for } p_{\mathrm{tree}} \le p < p_c \tag{13}$$

More precisely,

$$\ell(R, p) = \frac{R^{d_{\min}^{\mathrm{MST}}}}{v(p)}, \qquad v(p) \sim \xi(p)^{d_{\min}^{\mathrm{MST}} - d_{\min}^{\mathrm{perc}}} \sim (p_c - p)^{-\nu d_{\min}^{\mathrm{MST}} + \nu d_{\min}^{\mathrm{perc}}}, \tag{14}$$

Consider a certain quasitree $\mathcal{Q}$ and the set of trees $\mathcal{T}_i^{(\mathcal{Q})}$ which can originate from $\mathcal{Q}$ in the course of further destruction of bonds. Obviously, $\mathcal{Q}$ is a union of all such trees: $\mathcal{Q} = \cup_i \mathcal{T}_i^{(\mathcal{Q})}$. Now we introduce a graph

$$\overline{\mathcal{P}}(A, B)^{(\mathcal{Q})} = \cup_i \mathcal{P}_i(A, B)^{(\mathcal{Q})}, \tag{15}$$

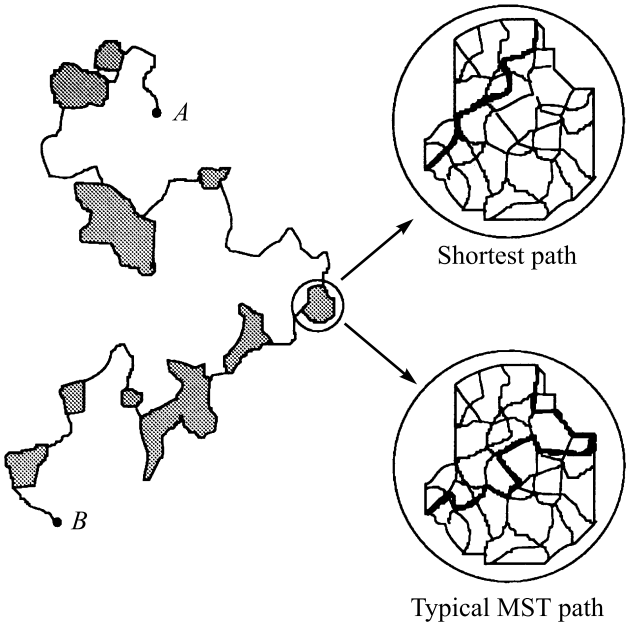

Fig.3. The path, connecting points A and B on the block graph; nontrivial blocks are shown as grey islands. The internal structure of a typical block is shown in inserts: a typical MST-path, traversing a block (lower panel); the shortest path across the block (upper panel)

which is the union of all paths $\mathcal{P}_i^{(\mathcal{Q})}$ leading from A to B in all trees $\mathcal{T}_i^{(\mathcal{Q})}$. It is easy to show that $\overline{\mathcal{P}}(A, B)^{(\mathcal{Q})}$ is precisely the path leading from A to B *in the block graph* (see Fig.3). The minimal (over the entire quasitree) path $\mathcal{P}(A, B)^{(\mathcal{Q})}$ leading from A to B is, obviously, the minimal path over the graph $\overline{\mathcal{P}}(A, B)^{(\mathcal{Q})}$. On nontrivial (containing more than one bond) blocks the path $\mathcal{P}(A, B)^{(\mathcal{Q})}$ is the shortest path that crosses the block; it may be considerably shorter than any individual MST-path. This consideration enables one to estimate the typical ratio of lengths for a piece of the minimal path $\mathcal{P}(A, B)^{(\mathcal{Q})}$ and the corresponding piece of the MST minimal path. We make such an estimate for the case when $p < p_c$ but $p_c - p \ll 1$ (i.e., for the vicinity of the phase transition). Having in mind that the typical block size is $\xi(p) \gg 1$, for a typical length MST-path crossing such a block we get $\ell_{\mathrm{MST}}(\xi) \sim \xi^{d_{\min}^{\mathrm{MST}}}$, while the *shortest* path traversing the block is the same as for the critical percolation: $\ell_{\mathrm{short}}(\xi) \sim \xi^{d_{\min}^{\mathrm{perc}}}$. As a result, we arrive at the estimate (14), which matches with (12) for the critical case $R \sim \xi$.

Since the above consideration is not quite rigorous, we have also undertaken numerical evaluation of $d_{\min}(p)$ in order to check the identity (13). Simulations did not show any variation of $d_{\min}$ with p for $p < p_c$ (see Fig.4).

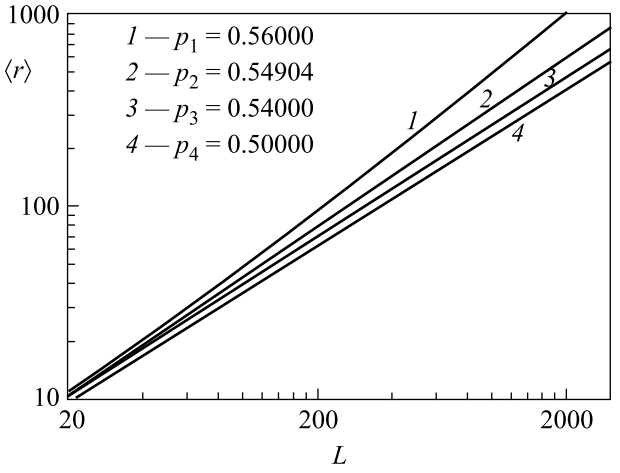

Fig.4. Root mean square euclidean displacement vs. chemical distance for different values of p, obtained via averaging over 1600 realizations of a 2048 × 2048 lattice with periodic boundary conditions. Asymptotic slopes correspond to $d_{\min}(p_c) = 1.13(1)$ and $d_{\min}(p_c - 0.01 = 0.54) = d_{\min}(p_{\mathrm{tree}}) = 1.22(1)$ in agreement with known results

In conclusion, we have demonstrated that the Self-Repairing Bond Percolation model undergoes a topological phase transition at a certain concentration p_c of remaining bonds. In the tree-like phase (for $p < p_c$) the

network, although being fully connected, has *no backbone* and hence zero conductivity. The corresponding graphs of bonds are "quasitrees": they contain only finite cycles (even for the infinite lattice). The properties of the statistical ensemble of quasitrees are similar to those of the Minimal Spanning Trees ensemble.

1. D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, Taylor and Fransis, London, 1994.
2. A. Bunde and S. Havlin, *Percolation I*, in *Fractals and Disordered Systems*, Eds. A.Bunde and S. Havlin, Springer, Berlin, 1996.
3. *Disorder and Granular Media*, Eds. D. Bideau and A. Hansen, North-Holland, Amsterdam, 1993.
4. G. Wypych, *Handbook of Material Weathering*, ChemTec Publishers, 2003; *The Effect of UV Light and Weather on Plastics and Elastomers*, PDL Staff, Plastics Design Library, 1994.
5. S. S. Manna and B. Subramanian, Phys. Rev. Lett. **76**, 3460 (1996).
6. R. Dobrin and P. M. Duxbury, Phys. Rev. Lett. **86**, 5076 (2001).
7. H. N. V. Temperley, E. H. Lieb, Proc. R. Soc. London **A322**, 251 (1971).
8. R. M. Ziff, S. Finch, and V. Adamchik, Phys. Rev. Lett. **79**, 3447 (1997).
9. F. Harary, *Graph Theory*, Addison-Wesley, 1969.
10. P. Grassberger, Physica **A262**, 251 (1999).
11. C. M. Newman and D. L. Stein, Phys. Rev. Lett. **72**, 2286 (1994).
12. M. Cieplak, A. Maritan, and J. R. Banavar, Phys. Rev. Lett. **76**, 3754 (1996).