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THE LEVEL SPACING DISTRIBUTION NEAR THE ANDERSON TRANSITION

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For a disordered system near the Anderson transition we show that the nearest-level-spacing distribution has the asymptotics $P(s) \propto \exp(-As^{2-\gamma})$ for $s \gg \langle s \rangle \equiv 1$ which is universal and intermediate between the Gaussian asymptotics in a metal and the Poisson in an insulator. (Here the critical exponent $0 < \gamma < 1$ and the numerical coefficient A depend only on the dimensionality $d > 2$). It is obtained by mapping the energy level distribution to the Gibbs distribution for a classical one-dimensional gas with a pairwise interaction. The interaction, consistent with the universal asymptotics of the two-level correlation function found previously, is proved to be the power-law repulsion with the exponent $-\gamma$.

It is well known [1, 2, 3] that a statistical description of energy levels of quantum disordered systems in the metallic phase is provided by the random matrix theory (RMT)[4]. Its most important characteristic is the repulsion between the energy levels at any scale. In the Anderson insulator phase, the energy levels are independent and described by the Poisson statistics, provided that the appropriate states are separated by the length exceeding the localization length.

It has been conjectured [5, 6] that a universal statistical description is possible also in the critical region in the vicinity of the Anderson transition between the two phases. The dimensional scaling estimation made in Ref.[5] for the variance $\langle N^2 \rangle - \langle N \rangle^2$ of the number of energy levels in a given energy interval, has suggested that it is proportional to $\langle N \rangle$, thus being different from the Poisson statistics only by a certain number. A different statistical characteristic, the nearest-level-spacing distribution, has been conjectured in Ref.[6] on the basis of numerical simulations to be some universal 'hybrid' of the Poisson distribution for large level spacings and the Wigner surmise (see below) for small spacing.

However, it has recently been analytically proved [7] that the universal statistics, exactly applicable near the Anderson transition point (mobility edge), is entirely new and drastically different from both the RMT and the Poisson limit. The variance of the number of levels in the energy interval $E \gg \Delta$ (centered at the Fermi energy) has been found as

$$\langle N^2 \rangle - \langle N \rangle^2 = \frac{a_d}{\beta} \left(\frac{E}{\Delta} \right)^{\gamma} \equiv \frac{a_d}{\beta} \langle N \rangle^{\gamma}, \qquad 0 < \gamma < 1. \tag{1}$$

Here Δ is the mean level spacing, $\langle \ldots \rangle$ denotes the ensemble averaging, the coefficient a_d and the critical exponent γ depend only on the dimensionality $d > 2$, and β is determined by the class of symmetry ($\beta = 1, 2$, or 4 for unitary, orthogonal, and symplectic ensembles, respectively [4]). Eq. (1) is exact at the mobility edge in the limit

$$L \to \infty, \qquad \langle N \rangle = \text{const} \gg 1. \tag{2}$$

In the same limit, the metallic phase is exactly described by the RMT and the insulator phase by the Poisson statistics.

Thus, the spectral rigidity does not disappear at the mobility edge (in contrast to the insulating phase where the levels are independent and the variance equals $\langle N \rangle$), but it is considerably weaker than in the metallic phase (where the fluctuations are suppressed and the variance is proportional to $\ln\langle N \rangle$). Then it is naturally to expect that the nearest-level-spacing distribution near the Anderson transition is also different from those both in the metal and insulating phases.

Indeed, we will show in this Letter that the asymptotics of this distribution at the mobility edge is given by

$$P(s) \propto \exp(-A_d \beta s^{2-\gamma}), \qquad s \equiv \omega/\Delta \gg 1, \tag{3}$$

where ω is a distance between adjacent levels, and A_d is some numerical factor depending only on the dimensionality d. This is drastically different from both the Poisson distribution, $P(s) = \exp(-s)$, and the exact Gaussian asymptotics in the metallic phase [4]

$$P(s) \sim \exp\left(-\tfrac{1}{16}\pi^2 \beta s^2\right) \tag{4}$$

Note that the famous Wigner surmise $P(s) = (\pi s/2)\exp(-\pi s^2/4)$ (for $\beta = 1$) describes this asymptotics only approximately [4].

Both the universal variance (1), and the asymptotics of the distribution (3) result from the exact asymptotics of the spectral density correlation function at the mobility edge

$$R(\omega) \equiv \frac{1}{\nu_0^2}\left\langle \nu(\varepsilon)\nu(\varepsilon') \right\rangle - 1 = -c_d \beta^{-1} |x - x'|^{-2+\gamma}, \qquad x \equiv \varepsilon/\Delta, \qquad |x - x'| \gg 1 \tag{5}$$

where $\nu(\varepsilon)$ is the exact density of states at the energy ε, ν_0 is the mean density of states, c_d is a positive number depending only on the dimensionality $d > 2$. The asymptotics (5) has been obtained in Ref.[7] by calculating all the diagrams (with accuracy up to a numerical coefficient) which turned out to be possible after taking into account the analytical properties of the diffusion propagator and certain scaling relations at the mobility edge.

To derive the announced result (3), we will use the effective "plasma model" as suggested by Dyson [8]. In such a model, the level distribution is mapped to the Gibbs distribution of a classical one-dimensional gas of fictitious "particles" with a repulsive pairwise interaction $f(|\varepsilon_i - \varepsilon_j|)$ in the presence of a confining potential $V(\varepsilon)$

$$\mathcal{P}(\{\varepsilon_n\}) = Z^{-1} \exp\left[-\beta \mathcal{H}(\{\varepsilon_n\})\right], \tag{6}$$

$$\mathcal{H}(\{\varepsilon_n\}) = \sum_{i<j} f(|\varepsilon_i - \varepsilon_j|) + \sum_i V(\varepsilon_i) \tag{7}$$

Here Z is the partition function and β plays a role of the inverse temperature. For $f(|\varepsilon_i - \varepsilon_j|) \equiv \ln|\varepsilon_i - \varepsilon_j|^{-1}$, Eqs. (6), (7) reproduce exactly the level distribution in the RMT, with β depending on the symmetry class as described after Eq. (1). The choice $V(\varepsilon_i) = \varepsilon_i^2$ for the confinement potential leads to the Gaussian ensembles but other choices are also possible [4, 9]. In the metallic phase such a description is exact for the energy separation $|\varepsilon_i - \varepsilon_j| < E_c \equiv \hbar/\tau_D$ where $\tau_D = L^2/D$ is the 'ergodic' time necessary for an electron to diffuse across the system. For $t < \tau_D$, i.e. $|\varepsilon_i - \varepsilon_j| > E_c$, the level statistics is completely different [3] from that of the RMT. However, it is also described with the Gibbs distribution (6), (7), albeit with the pairwise interaction f having the power-law asymptotics [10].

At the mobility edge $E_c \sim \Delta$ ($g = E_c/\Delta$ is a dimensionless conductance). Therefore, the energy separation of a few Δ is already outside of the RMT region of validity. We will show that the asymptotics of the correlation function (5) at the mobility edge is described correctly by the Gibbs model with the power-law interaction

$$f(|x-x'|) = \frac{1-\gamma}{2\pi c_d} \cot(\pi\gamma/2)\,|x-x'|^{-\gamma}, \qquad 0 < \gamma < 1, \quad x \equiv \varepsilon/\Delta. \tag{8}$$

(Naturally, this interaction is different from that in Ref.[10], where $\gamma > 1$, which describes the *nonuniversal* level statistics in the metallic phase at the scale $\omega \gg E_c$.) Before proving this, we will show how the form of the pairwise interaction governs the asymptotics of $P(s)$.

The distribution $P(s)$ describes the probability to find the nearest adjacent level at the distance $s = \omega/\Delta$ from a given one. It is equivalent to the probability to find a "gap" of the width s (i.e. region that contains no "particles") in the Gibbs model. This probability is obtained [4] from Eq. (6) as

$$P(s) = \exp\left[-\beta(F_s - F_0)\right] \tag{9}$$

where F_s is the free energy of the one-dimensional gas (7) distributed along the straight line with the gap s around its center. For $s \gg 1$, one introduces a continuous density $\rho_s(x)$ to describe such a distribution. Then, in the mean-field approximation (MFA) F_s may be expressed as the functional

$$o_s(x)\rho_s(x')f(|x-x'|) + \int_{|x|\geq\frac{s}{2}} dx\, \rho_s(x)V(x), \tag{10}$$

where $\rho_s(x)$ obeys the mean-field (MF) equation

$$\int_{|x'|\geq\frac{s}{2}} dx'\, \rho_s(x')f(|x-x'|) = -V(x) - \mu_s, \qquad |x| \geq s/2. \tag{11}$$

Here μ_s arises from the "particles" number conservation (corresponding to the level number conservation in the original quantum disordered system),

$$\int_{|x|\geq\frac{s}{2}} dx\,\rho_s(x)dx = \int_{-\infty}^{\infty} \rho_0(x) = \mathcal{N}. \tag{12}$$

Taking $s = 0$ in Eqs. (10) and (11), one finds the density $\rho_0(x)$ and free energy F_0 for a homogeneous distribution.

Equations similar to (10) and (11) have been derived for a circular ensemble (a classical gas with the $\log|x-x'|^{-1}$ interaction confined to a circular wire) by Dyson [8] (see also Ref.[4]) who has also found the corrections to the MF solution (allowing for the entropy term added to the functional (10) and for the discreteness of the original distribution) lead to a linear in s contribution to the difference $F_s - F_0$. For $s \gg 1$ this contribution is small compared to the leading quadratic term, Eq. (4). We will consider only the terms leading in the $s \gg 1$ limit which are described by the MFA.

For the circular ensemble [8] there was no need in the confining potential $V(x)$. We use the linear ensemble that is more convenient for relating the interaction $f(|\varepsilon_i - \varepsilon_j|)$ to the correlation function (5). Equation (11) with the weakly singular kernel (8) can be solved for any $V(x)$. This exact solution shows a strong dependence of both ρ_s and ρ_0 (and thus F_s and F_0) on $V(x)$ (see Eq. (17) for ρ_0). However, it is easy to show that the difference $F_s - F_0$, and thus the distribution (9), does not depend on V for $s \gg 1$ in the limit $\mathcal{N} \to \infty$. Furthermore, the asymptotics of this universal distribution may be found, with accuracy up to a numerical coefficient, without knowing the explicit solution to Eq. (11).

The explicit dependence of $F_s - F_0$ on $V(x)$, Eq. (10), is excluded straightforwardly with the help of Eqs. (11), (12). Then, after some transformations using the fact that the change in the "chemical potential" due to the gap formation $\mu_s - \mu_0 \sim s/\mathcal{N} \ll 1$, one finds with the accuracy up to $s/\mathcal{N}$ that

$$\begin{aligned} F_s - F_0 = &-\frac{1}{2}\int_{|x|\geq\frac{s}{2}} dx \int_{|x'|\geq\frac{s}{2}} dx'\,\delta\rho_x\delta\rho(x')f(|x-x'|)+ \\ &+\frac{1}{2}\int_{-s/2}^{s/2} dx \int_{-s/2}^{s/2} dx'\rho_0(x)\rho_0(x')f(|x-x'|) \end{aligned} \tag{13}$$

where $\delta\rho(x) = \rho_s(x) - \rho_0(x)$ decreases rapidly for $x \gg s$. The function $\delta\rho(x)$ obeys the MF equation:

$$\int_{|x'|\geq\frac{s}{2}} dx'\,\delta\rho(x')\,f(|x-x'|) = \int_{-s/2}^{s/2} dx'\rho_0(x')\,f(|x-x'|), \qquad |x| \geq s/2, \tag{14}$$

which follows from Eq. (11), if one neglects the small term $\mu_s - \mu_0 \sim s/\mathcal{N}$. The homogeneous level density $\rho_0(x)$ in Eqs. (13), (14) still depends on $V(x)$, Eq. (11). However, for $|x| < s/2 \ll \mathcal{N}$, this dependence is negligible, and in the limit $\mathcal{N} \to \infty$ one finds $\rho_0 = 1$ (in units of $1/\Delta$). Now it is clearly seen from Eqs. (13) and (14) that the quantity $F_s - F_0 = -\beta^{-1}\ln P(s)$ is determined by the interaction $f(|x-x'|)$ only.

Equations (13) and (14) are valid for an arbitrary long-ranged interaction $f(|x-x'|)$. For the case of the power-law interaction (8) (with $0<\gamma<1$), one may rescale $x \to sx$ and $x' \to sx'$ to find that the solution of Eq. (14) (with $\rho_0 =$const) has the form $\delta\rho(x) = \varphi(x/s)$, where $\varphi(x)$ is a universal function independent of s. Substituting this solution to Eq. (13) we arrive after the same rescaling at:

$$F_s - F_0 = -\beta^{-1} \ln P(s) = A_\gamma s^{2-\gamma} \tag{15}$$

where the universal constant A_γ depends only on the power of the interaction. Calculating this constant for the limiting case ($\gamma = 0$) of the logarithmic interaction in Eqs. (13), (14) one reproduces the asymptotics (4) known from the RMT [4] which leads to the announced result Eq. (3).

To prove that the interaction (8) reproduces the correlation function (5), we use the relationship [9]:

$$R(x,x') = -\beta^{-1} \frac{\delta\rho_0(x)}{\delta V(x')}. \tag{16}$$

For the interaction $f = a_\gamma |x-x'|^{-\gamma}$ and an arbitrary $v(x) \equiv V(x) + \mu_0$, the solution to Eq. (11) with $s = 0$ is found, using the methods described in Ref.[11], as follows:

$$\rho_0(x) = \frac{\cos^2(\pi\gamma/2)(x+D)^{\frac{\gamma-1}{2}}}{\pi^2 a_\gamma} B(\gamma, \tfrac{1-\gamma}{2}) \times$$

$$\times \frac{d}{dx}\left\{ \int_x^D dt\,(t+D)^{1-\gamma}(t-x)^{\frac{\gamma-1}{2}} \frac{d}{dt} \int_{-D}^{t} d\tau\,(\tau+D)^{\frac{\gamma-1}{2}}(t-\tau)^{\frac{\gamma-1}{2}}\, v(\tau) \right\}, \tag{17}$$

Here B is the Euler function, D is the band edge that may be found from Eq. (12) and tends to infinity when $\mathcal{N} \to \infty$. Taking the variational derivative (16), i.e. substituting $-\beta^{-1}\delta(\tau - x')$ for $v(\tau)$ in Eq. (17), one finds in the limit $D \to \infty$:

$$R(x,x') = -\beta^{-1} \frac{1-\gamma}{2\pi a_\gamma} \cot\left(\frac{\pi\gamma}{2}\right) |x-x'|^{\gamma-2}. \tag{18}$$

So the Gibbs model with the power-law interaction results in the asymptotics (5) of the correlation function. Comparing Eqs. (5) and (18), we obtain Eq. (8).

Note that for all the three universal statistics, in the metal and insulating phases and at the mobility edge, a simple relation holds between the variance of the level number fluctuations in the limit (2), and the asymptotics of the nearest-level-spacing distribution. Namely, if the variance proportional to $\langle N \rangle^\gamma$, then $-\ln P(s) \propto s^{2-\gamma}$. The linear in $\langle N \rangle$ variance is forbidden at the mobility edge [7] by the exact sum rule that is due to the conservation of the total number of states $\mathcal{N}$. Therefore, the Poisson (i.e. linear in s) asymptotics of $P(s)$ is equally forbidden at the mobility edge. Finally, following Ref. [6], we note that for $s \ll 1$ the distribution $P(s)$ shows at the mobility edge the same behaviour as in the metallic phase, $P(s) \sim s^\beta$, which follows from the general symmetry theorem proved by Dyson [8]. Then, the whole distribution could be described by the following surmise:

$$P(s) = Bs^\beta \exp\left(-A_d \beta s^{2-\gamma}\right) \tag{19}$$

where B is found from the normalization conditions.

1. L.P.Gor'kov and G.M.Eliashberg, Zh. Eksp. Teor. Fiz. **48**, 1407 (1965) [Sov. Phys. JETP **21**, 940 (1965)].
2. K.B.Efetov, Adv.Phys. **32**, 53 (1983).
3. B.L.Altshuler and B.I.Shklovskii, Zh. Eksp. Teor. Fiz. **91**, 220 (1986) [Sov. Phys. JETP **64**, 127 (1986)].
4. M. L. Mehta, Random matrices (Academic Press, Boston, 1991).
5. B.L.Altshuler, I.K.Zharekeshev, S.A.Kotochigova, and B.I.Shklovskii, Zh. Eksp. Teor. Fiz. **94**, 343 (1988) [Sov. Phys. JETP **67**, 625 (1988)].
6. B.I.Shklovskii, B.Shapiro, B.R.Sears et al., Phys. Rev. B **47**, 11487 (1993).
7. V.E.Kravtsov, I.V.Lerner, B.L.Altshuler, and A.G.Aronov, Phys. Rev. Lett. to be published (1994).
8. F.J.Dyson, J. Math. Phys. **3**, 140 (1962).
9. C.W.J.Beenakker, Phys. Rev. Lett. **70**, 1155 (1993).
10. R.A.Jalabert, J.-L.Pichard, and C.W.J.Beenakker, Europhys. Lett **24**, 1 (1993).
11. D.Porter and D.S.G.Stirling, *Integral equations (Cambridge University Press, 1990).*