ON QUANTIZATION OF SYSTEMS BASED ON THE ODD POISSON BRACKET

V.A.Soroka[1]

Kharkov Institute of Physics and Technology
310108 Kharkov, Ukraine

New quantum representations of the odd (with respect to the Grassmann grading) Poisson bracket are obtained, some of them, as demonstrated by the simplest example, are responsible for the quantization of the classical Hamilton's dynamics based on the odd Poisson bracket.

1. A few years ago the prescription [1] for the canonical quantization of the odd Poisson bracket (the last has naturally appeared in the Batalin-Vilkovisky scheme [2] for the quantization of the gauge theories) was suggested, and several odd-bracket quantum representations for the canonical variables were also obtained. In contrast with the even Poisson bracket case, some of the odd-bracket quantum representations turned out to be no equivalent [3]. Later, it has been revealed that odd Poisson bracket is responsible for the description of the dynamics of some Hamilton systems [4]. Namely, for the systems having an equal number of pairs of even and odd (relative to the Grassmann grading) phase coordinates it was proved that Hamilton's equations of motion obtained by means of the even Poisson bracket with the help of the even Hamiltonian can be reproduced by the odd bracket using the equivalent odd Hamiltonian. However, the direct connection of the odd-bracket quantum representations for the canonical variables with the quantization of the classical Hamilton dynamics based on the odd Poisson bracket has not been formulated explicitly until now. To this end, having two equivalent ways of the classical dynamics description for the above-mentioned systems [4], one can try to find such odd-bracket quantum representations, which with the use of the classically equivalent odd Hamiltonian will provide the quantum description of the systems, coinciding with that obtained from the corresponding even Hamiltonian with the use of the even-bracket quantum representations. In other words, the equivalence of the description of dynamics with the brackets of different Grassmann parities can be extended from a classical level to the quantum one.

In the present letter, new odd-bracket quantum representations, extending those obtained in [1, 3], are introduced. At the end of the paper, the simplest example of the supersymmetric one-dimensional oscillator demonstrates that among these representations are just the ones relevant to the quantization of the classical Hamilton systems, whose dynamics is formulated by means of the odd bracket with the help of the odd Hamiltonian.

2. First, we recall the necessary properties of various graded Poisson brackets. The even and odd brackets in terms of the real even $y_i = (q^a, p_a)$ and odd $\eta^i = \theta^\alpha$ canonical variables have, respectively, the form

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[1] E-MAIL: KFTI%KFTI.KHARKOV.UA@RELAY.USSR.EU.NET

$$\{A, B\}_o = A \left[\sum_a^n \left(\overleftarrow{\partial}_{q^a} \overrightarrow{\partial}_{p_a} - \overleftarrow{\partial}_{p_a} \overrightarrow{\partial}_{q^a} \right) - i \sum_{\alpha=1}^{2m} \overleftarrow{\partial}_{\theta^\alpha} \overrightarrow{\partial}_{\theta^\alpha} \right] B \; ; \tag{1}$$

$$\{A, B\}_1 = A \sum_{i=1}^{N} \left(\overleftarrow{\partial}_{y_i} \overrightarrow{\partial}_{\eta^i} - \overleftarrow{\partial}_{\eta^i} \overrightarrow{\partial}_{y_i} \right) B \; , \tag{2}$$

where $g(A)$ is the Grassmann grading of the quantity A, $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ are the right and left derivatives, and the notation $\partial_x = \frac{\partial}{\partial x}$ is introduced. By introducing apart from the Grassmann grading $g(A)$ of any quantity A its corresponding bracket grading $g_\epsilon(A) = g(A) + \epsilon \pmod 2$ ($\epsilon = 0, 1$), the grading and symmetry properties, the Jacobi identities and the Leibnitz rule are uniformly expressed for the both brackets (1,2) as

$$g_\epsilon(\{A, B\}_\epsilon) = g_\epsilon(A) + g_\epsilon(B) \pmod 2 \; , \tag{3}$$

$$\{A, B\}_\epsilon = -(-1)^{g_\epsilon(A) g_\epsilon(B)} \{B, A\}_\epsilon \; , \tag{4}$$

$$\sum_{(ABC)} (-1)^{g_\epsilon(A) g_\epsilon(C)} \{A, \{B, C\}_\epsilon\}_\epsilon = 0 \; , \tag{5}$$

$$\{A, BC\}_\epsilon = \{A, B\}_\epsilon \, C + (-1)^{g_\epsilon(A) g(B)} \, B\{A, C\}_\epsilon \; , \tag{6}$$

where (3)–(5) have the shape of the Lie superalgebra relations in their canonical form [5] with $g_\epsilon(A)$ being the canonical grading for the corresponding bracket.

3. The procedure of the odd-bracket canonical quantization given in [1, 3] resides in splitting all the canonical variables into two sets, in the division of all the functions dependent on the canonical variables into classes, and in the introduction of the quantum multiplication $*$, which is either the common product or the bracket composition, in dependence on what the classes the co-factors belong to. Under this, one of the classes has to contain the normalized wave functions, and the result of the multiplication $*$ for any quantity on the wave function Ψ must belong to the class containing Ψ. This procedure is the generalization on the odd bracket case of the canonical quantization rules for the usual Poisson bracket $\{\ldots,\ldots\}_{Pois.}$, which, for example, in the coordinate representation for the canonical variables q and p is defined as

$$q \; * \Psi(q) = q\Psi(q) \; , \qquad p \; * \Psi(q) = i\hbar\{p, \Psi(q)\}_{Pois.} = -i\hbar\frac{\partial \Psi}{\partial q} \; ,$$

where $\Psi(q)$ is the normalized wave function depending on the coordinate q.

In [1, 3] two nonequivalent odd-bracket quantum representations for the canonical variables were obtained by using two different ways of the function division. But these ways do not exhaust all the possibilities. In the present paper a more general way of the division is proposed, which contains as the limiting cases the ones given in [1, 3].

Let us build quantum representations for an arbitrary graded bracket under its canonical quantization. To this end, all canonical variables are split into two equal in the number sets, so that none of them should contain the pairs of canonical conjugates. Note that to make such a splitting possible for the even bracket (1), the transition has to be done from the real canonical self-conjugate odd variables

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to some pairs of odd variables, which simultaneously are complex and canonical conjugate to each other. Composing from the integer degrees of the variables from the one set (we call it the first set) the monomials of the odd $2s+1$ and even $2s$ uniformity degrees and multiplying them by the arbitrary functions dependent on the variables from the other (second) set, we thus divide all the functions of the canonical variables into the classes designated as $\overset{\epsilon}{O}_s$ and $\overset{\epsilon}{E}_s$, respectively. For instance, in the general case the odd-bracket canonical variables can be split, so that the first set would contain the even $\overset{\circ}{y}_i$ $(i=1,\ldots,n \le N)$ and odd $\eta^{n+\alpha}$ $(\alpha = 1,\ldots,N-n)$ variables, while the second set would involve the rest variables. Then the classes of the functions obtained under this splitting have the form

$$\overset{1}{O}_s = \left(y_i, \eta^{n+\alpha}\right)^{2s+1} f\left(\eta^i, y_{n+\alpha}\right) \quad ; \qquad \overset{1}{E}_s = \left(y_i, \eta^{n+\alpha}\right)^{2s} f\left(\eta^i, y_{n+\alpha}\right) \quad ,$$

where the factors before the arbitrary function $f\left(\eta^i, y_{n+\alpha}\right)$ denote the monomials having the uniformity degrees indicated in the exponents. These classes satisfy the corresponding bracket relations

$$\{\overset{\epsilon}{O}_s, \overset{\epsilon}{O}_{s'}\}_\epsilon = \overset{\epsilon}{O}_{s+s'} \quad ; \qquad \{\overset{\epsilon}{O}_s, \overset{\epsilon}{E}_{s'}\}_\epsilon = \overset{\epsilon}{E}_{s+s'} \quad ; \qquad \{\overset{\epsilon}{E}_s, \overset{\epsilon}{E}_{s'}\}_\epsilon = \overset{\epsilon}{O}_{s+s'-1} \quad , \qquad (7)$$

and the relations of the ordinary Grassmann multiplication

$$\overset{\epsilon}{O}_s \cdot \overset{\epsilon}{O}_{s'} = \overset{\epsilon}{E}_{s+s'+1} \quad ; \qquad \overset{\epsilon}{O}_s \cdot \overset{\epsilon}{E}_{s'} = \overset{\epsilon}{O}_{s+s'} \quad ; \qquad \overset{\epsilon}{E}_s \cdot \overset{\epsilon}{E}_{s'} = \overset{\epsilon}{E}_{s+s'} \quad . \qquad (8)$$

It follows from (7),(8), that $\overset{\epsilon}{O} = \{\overset{\epsilon}{O}_s\}$ and $\overset{\epsilon}{E} = \{\overset{\epsilon}{E}_s\}$ form a superalgebra with respect to the addition and the quantum multiplication $*_\epsilon$ $(\epsilon = 0, 1)$ defined for the corresponding bracket as

$$\overset{\epsilon'}{O} *_\epsilon \overset{\epsilon''}{O} = \{\overset{\epsilon'}{O}, \overset{\epsilon''}{O}\}_\epsilon \in \overset{\epsilon}{O} \quad ; \quad \overset{\epsilon'}{O} *_\epsilon \overset{\epsilon''}{E} = \{\overset{\epsilon'}{O}, \overset{\epsilon''}{E}\}_\epsilon \in \overset{\epsilon}{E} \quad ; \quad \overset{\epsilon'}{E} *_\epsilon \overset{\epsilon''}{E} = \overset{\epsilon'}{E} \cdot \overset{\epsilon''}{E} \in \overset{\epsilon}{E} \quad , \tag{9}$$

where $\overset{\epsilon'}{O}, \overset{\epsilon''}{O} \in \overset{\epsilon}{O}$ and $\overset{\epsilon'}{E}, \overset{\epsilon''}{E} \in \overset{\epsilon}{E}$. Note, that the classes $\overset{\epsilon}{O}_0$ and $\overset{\epsilon}{E}_0$ form the sub-superalgebra. In terms of the quantum grading $q_\epsilon(A)$ of any quantity A

$$q_\epsilon(A) = \begin{cases} g_\epsilon(A), & \text{for } A \in \overset{\epsilon}{O}; \\ g(A), & \text{for } A \in \overset{\epsilon}{E}, \end{cases}$$

introduced for the appropriate bracket, the grading and symmetry properties of the quantum multiplication $*_\epsilon$, arising from the corresponding properties for the bracket (3,4) and Grassmann composition of any two quantities A and B, are uniformly written as

$$q_\epsilon(A *_\epsilon B) = q_\epsilon(A) + q_\epsilon(B)(\text{mod}2) \tag{10a}$$

$$\overset{\epsilon'}{O} *_\epsilon \overset{\epsilon''}{O} = -(-1)^{g_\epsilon(\overset{\epsilon'}{O})g_\epsilon(\overset{\epsilon''}{O})} \overset{\epsilon''}{O} *_\epsilon \overset{\epsilon'}{O} \quad , \tag{10b}$$

$$\overset{\epsilon'}{E} *_\epsilon \overset{\epsilon''}{E} = (-1)^{g_\epsilon(\overset{\epsilon'}{E})g_\epsilon(\overset{\epsilon''}{E})} \overset{\epsilon''}{E} *_\epsilon \overset{\epsilon'}{E} \quad , \tag{10c}$$

With the use of the quantum multiplication $*_\epsilon$ and the quantum grading q_ϵ, let us define for any two quantities A, B the quantum bracket ((anti)commutator)

$[A, B]_\epsilon$ (under its action on the wave function Ψ that is considered to belong to the class E [1, 3]) in the form

$$[A, B]_\epsilon *_\epsilon \Psi = A *_\epsilon (B *_\epsilon \Psi) - (-1)^{g_\epsilon(A)g_\epsilon(B)} B *_\epsilon (A *_\epsilon \Psi) . \tag{11}$$

If $A, B \in \overset{\epsilon}{E}$, then, due to (10c), the quantum bracket between them equals zero. In particular, the wave functions are (anti)commutative. If A or both of the quantities A and B belong to the class $\overset{\epsilon}{O}$, then in the first case, due to the Leibnitz rule (6), and in the second one, because of the Jacobi identities (5), the relation follows from the definitions (10) and (11)

$$[A, B]_\epsilon *_\epsilon \Psi = \{A, B\}_\epsilon *_\epsilon \Psi = (A *_\epsilon B) *_\epsilon \Psi ,$$

that establishes the connection between the classical and quantum brackets of the corresponding Grassmann parity. Note, that the quantization procedure also admits the reduction to $O_o \cup E_o$.

The grading q_ϵ determines the symmetry properties of the quantum bracket (11). Under above-mentioned splitting of the odd-bracket canonical variables into two sets, the grading q_1 equals unity for the variables $y_i \in \overset{1}{O}$, $\eta^i \in \overset{1}{E}$ ($i = 1, \ldots, n \leq N$) and equal to zero for the rest canonical variables $y_{n+\alpha} \in \overset{1}{E}$, $\eta^{n+\alpha} \in \overset{1}{O}$ ($\alpha = 1, \ldots, N - n$). Therefore, in this case the quantum odd bracket is represented with the anticommutators between the quantities y_i, η^i and with the commutators for the remaining relations of the canonical variables. If the roles of the first and the second sets of the canonical variables change, then the quantum bracket is represented with the anticommutators between $y_{n+\alpha}, \eta^{n+\alpha}$ and with the commutators in the other relations. In [1, 3]) the odd-bracket quantum representations were obtained for the cases $n = 0, N$, containing, respectively, only commutators or anticommutators.

4. As the simplest example of using of the odd-bracket quantum representations under the quantization of the classical systems based on the odd bracket, let us consider the one-dimensional supersymmetric oscillator, whose phase superspace x^A contains a pair of even q, p and a pair of odd η^1, η^2 real canonical coordinates. In terms of more suitable complex coordinates $z = (p - iq)/\sqrt{2}$, $\eta = (\eta^1 - i\eta)/\sqrt{2}$ and their complex conjugates $\bar{z}, \bar{\eta}$, the even bracket is written as

$$\{A, B\}_0 = iA \left[\overset{\leftarrow}{\partial_{\bar z}} \overset{\rightarrow}{\partial_z} - \overset{\leftarrow}{\partial_z} \overset{\rightarrow}{\partial_{\bar z}} - \left(\overset{\leftarrow}{\partial_{\bar \eta}} \overset{\rightarrow}{\partial_\eta} - \overset{\leftarrow}{\partial_\eta} \overset{\rightarrow}{\partial_{\bar \eta}} \right) \right] B \tag{12}$$

and the even Hamiltonian H, the supercharges Q_1, Q_2 and the fermionic charge F have the forms

$$H = z\bar{z} + \bar{\eta}\eta \; ; \quad Q_1 = \bar{z}\eta + z\bar{\eta} \; ; \quad Q_2 = i(\bar{z}\eta - z\bar{\eta}) \; ; \quad F = \eta\bar{\eta} . \tag{13}$$

The odd Hamiltonian H and the appropriate odd bracket, which reproduce the same Hamilton equations of motion, as those resulting from (12) with the even Hamiltonian H (13), i.e., which satisfy the condition [4, 6]

$$\frac{dx^A}{dt} = \{x^A, H\}_0 = \{x^A, H\}_1 \; . \tag{14}$$

(t is the time) can be taken as $\bar{H} = Q_1$ and

$$\{A, B\}_1 = iA \left(\overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_\eta - \overleftarrow{\partial}_\eta \overrightarrow{\partial}_{\bar{z}} + \overleftarrow{\partial}_{\bar{\eta}} \overrightarrow{\partial}_z - \overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{\eta}} \right) B \ . \tag{15}$$

The complex variables have the advantage over the real ones, because with their use the splitting of the canonical variables into two sets $\bar{z}, \bar{\eta}$ and z, η satisfies simultaneously the requirements necessary for the quantization both of the brackets (12,15). Besides, any of the vector fields $\overset{\epsilon}{X}_{A_i} = -i\{A_i, \ldots\}_\epsilon$ for the quantities $\{A_i\} = (H, Q_1, Q_2, F)$, describing the dynamics and the symmetry of the system under consideration, is split into the sum of two differential operators dependent on either $\bar{z}, \bar{\eta}$ or z, η . For instance, from (12)-(15) we have

$$\overset{0}{X}_H = \overset{1}{X}_H = z\partial_z + \eta\partial_\eta - \bar{z}\partial_{\bar{z}} - \bar{\eta}\partial_{\bar{\eta}} \ . \tag{16}$$

The diagonalization does not take place in terms of the variables $x^A = (q, p; \eta^1, \eta^2)$.

In accordance with the above-mentioned splitting of the complex variables, we can perform one of the two possible divisions all of the functions into the classes, which are common for both of the brackets (12),(15), playing a crucial role under their canonical quantization and leading to the same quantum dynamics for the system under consideration. If $\bar{z}, \bar{\eta}$ are attributed to the first set, then the corresponding function division is

$$\overset{\epsilon}{O}_s = (\bar{z}\bar{\eta})^{2s+1} f(z, \eta) \ ; \qquad \overset{\epsilon}{E}_s = (\bar{z}\bar{\eta})^{2s} f(z, \eta) \ .$$

If we restrict ourselves to the classes O_o and E_o, then $\Psi \in E_o$ and depends only on z, η and $A_i \in O_o$. According to the definition (9), the results of the quantum multiplications $*_1$ and $*_0$ of $z, \eta \in E_o$ and $\bar{z}, \bar{\eta} \in O_o$ on the wave function Ψ are

$$z *_1 \Psi = z *_0 \Psi = z \cdot \Psi \ ; \qquad \bar{\eta} *_1 \Psi = \bar{z} *_0 \Psi = \partial_z \Psi \ ;$$

$$\tag{17}$$

$$\eta *_1 \Psi = \eta *_0 \Psi = \eta \cdot \Psi \ ; \qquad \bar{z} *_1 \Psi = -\bar{\eta} *_0 \Psi = \partial_\eta \Psi \ .$$

The positive definite scalar product of the wave functions $\Psi_1(z, \eta)$ and $\Psi_2(z, \eta)$ can be determined in the form [7]

$$(\Psi_1, \Psi_2) = \frac{1}{\pi} \int \exp[-(|z|^2 + \bar{\theta}\eta)] \ \Psi_1(z, \eta) \ [\Psi_2(z, \theta)]^+ d\bar{\theta} \, d\eta \, d(Rez) \, d(Imz) \ , \tag{18}$$

where θ is the auxiliary complex Grassmann quantity anticommuting with η, and the integration over the real and imaginary components of z is performed in the limits $(-\infty, \infty)$. It is easy to see that with respect to the scalar product (18) the pairs of the canonical variables, being Hermitian conjugated to each other under the multiplication $*_1$, are $z, \bar{\eta}$ and $\bar{z}, \eta$, but under $*_0$ are $z, \bar{z}$ and $\eta, -\bar{\eta}$.

In order to have the action of the Hamiltonian operator, obtained from the system quantization, on the wave function, we need, as it is well known, to replace the canonical variables in the classical Hamiltonian by the respective operators or, which is the same, to define their action with the help of the corresponding quantum multiplication $*_\epsilon$. In this connection, in view of (16),(17), we see that the self-consistent quantum Hamilton operators in the even and odd cases, being

in agreement with the classical expressions (13) for the equivalent Hamiltonians H and $\bar{H}$ and giving the same result at the action on $\Psi(z, \eta)$, will be respectively

$$H *_{_0} \Psi = z *_{_0} (\bar{z} *_{_0} \Psi) - \eta *_{_0} (\bar{\eta} *_{_0} \Psi) \; ; \qquad (19)$$

$$\bar{H} *_{_1} \Psi = z *_{_1} (\bar{\eta} *_{_1} \Psi) + \eta *_{_1} (\bar{z} *_{_1} \Psi) \; . \qquad (20)$$

The Hamiltonians (19),(20) are Hermitian relative to the scalar product (18) and both, due to (17), are reduced to the Hamilton operator for the one-dimensional supersymmetric oscillator $H = a^+ a + b^+ b$ expressed in terms of the creation and annihilation operators for the bosons $a^+ = z$, $a = \partial_z$ and fermions $b^+ = \eta$, $b = \partial_\eta$ respectively, in the Fock-Bargmann representation (see, for example [8]). The normalized with respect to (18) eigenfunctions $\Psi_{k,n}(z, \eta)$ of the Hamiltonians (19),(20), corresponding to energy eigenvalues $E_{k,n} = k + n$ $(k = 0, 1; n = 0, 1, \ldots, \infty)$ have the form

$$\Psi_{k,n}(z, \eta) = \frac{1}{\sqrt{n!}} \left(\eta *_{_\epsilon} \right)^k \left(z *_{_\epsilon} \right)^n 1 \; .$$

Note, that another equivalent representation of the quantum supersymmetric oscillator can be obtained, if the canonical variables z, η are chosen as the first set.

5. Thus, we have demonstrated that the use of the quantum representations found for the odd bracket leads to the self-consistent quantization of the classical Hamilton systems based on this bracket. We should apparently expect that these representations are also applicable for the quantization of more complicated classical systems with the odd bracket.

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