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SCALING OF CORRELATION FUNCTIONS OF VELOCITY GRADIENTS IN HYDRODYNAMIC TURBULENCE

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As is demonstrated in [2, 3] in the limit of the infinitly large Reynolds number the correlation functions of the velocity predicted by Kolmogorov's 1941 theory (K41) are actually solutions of diagrammatic equations. Here we demonstrate that correlation functions of the velocity derivatives $\nabla_\alpha v_\beta$ should possess scaling exponents which has no relation to K41 dimensional estimates, this phenomenon will be refered to as anomalous scaling. This result is proved on the diagrammatic language: we have extracted series of logarithmically diverging diagrams summation of which leads to renormalization of normal K41 dimensions. For a description of the scaling of various functions of $\nabla_\alpha v_\beta$ an infinite set of primary fields O_n with independent scaling exponents Δ_n can be introduced. Symmetry reasons enable us to predict relations between scaling of different correlation functions. Besides we formulate restrictions imposed on the structure of correlation functions due to the incompressibility condition. We also propose some tests enabling ones to check experimentally the conformal symmetry of the turbulent correlation functions. Further we demonstrate that the anomalous scaling behavior should reveal in the asymptotic behavior of correlations function of velocity differences and propose a way to obtain the anomalous exponents from the experiment.

The theory of turbulence is the theory of strongly fluctuating hydrodynamic motion. Systems with strong fluctuations are examined both in quantum field theory and in condensed matter physics, e.g., in treating second order phase transitions. It is known that adequate tools of theoretical investigation of strong fluctuating systems are based upon functional integration methods, on different versions of the diagrammatic technique and on related methods. Therefore a consistent theory of turbulence should also be constructed in these terms.

The diagram technique for the problem of turbulence was developed by Wyld [1], who started from the Navier-Stokes equation with a pumping force. The Wyld technique enables one to represent any correlation function characterizing the turbulent flow as a series over the nonlinear interaction. Unfortunately infrared divergences appear in the technique. To avoid the divergences we will make use of the quasi-Lagrangian (qL) variables. The perturbation theory of the Wyld type in qL variables was developed by Belinicher and L'vov [2] (see also the review [3]).

The Wyld diagrammatic expansion is formulated in terms of propagators G and F and vertices determined by the nonlinear term of the Navier-Stokes equation. The G-function is the linear susceptibility determining the average value $\langle v_\alpha \rangle$ of the velocity $\mathbf{v}$ which arises as a response to the nonzero average $\langle \tilde{f}_\alpha \rangle$:

$$G_{\alpha\beta}(t, \mathbf{r}_1, \mathbf{r}_2) = -i\delta\langle v_\alpha(t, \mathbf{r}_1)\rangle / \delta\langle f_\beta(0, \mathbf{r}_2)\rangle \,, \tag{1}$$

where f is the pumping force. The F-function is the pair correlation function of $\mathbf{v}$:

$$F_{\alpha\beta}(t, \mathbf{r}_1, \mathbf{r}_2) = \langle v_\alpha(0, \mathbf{r}_1) v_\beta(t, \mathbf{r}_2) \rangle . \tag{2}$$

Note that the propagators G and F in qL variables depend separately on coordinates of the points $\mathbf{r}_1$, $\mathbf{r}_2$. Besides the simultaneous correlation function $F(t = 0, \mathbf{r}_1, \mathbf{r}_2)$ coinciding with the simultaneous correlation function of Eulerian velocities depends only on the difference $\mathbf{r}_1 - \mathbf{r}_2$.

To establish the behavior of (1), (2) we can utilize the dimensional estimations by Kolmogorov and Obukhov [4, 5]. For the pair correlation function (2) we obtain

$$F(t, \mathbf{r}_1, \mathbf{r}_2) \sim (\bar{\varepsilon} R)^{2/3} , \tag{3}$$

where R is the characteristic scale and $\bar{\varepsilon}$ is the average value of the energy dissipation rate per unit mass

$$\varepsilon = (\nu/2)(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha)^2 \ . \tag{4}$$

The Green's function also possess the scaling behavior with

$$G(t, \mathbf{r}_1, \mathbf{r}_2) \sim R^{-3} . \tag{5}$$

The question arises: can such scaling behavior be obtained as a solution of diagrammatic equations?

To answer this question one should first reformulate the diagram technique in terms of the bare vertices but of the dressed propagators F and G. Then one can easily check that the scaling behavior of F and G determined by the estimates (3), (5) is reproduced in any order of the perturbation theory. But this is not sufficient to justify the assertion that F and G actually possess such scaling behavior. The reason for this was long ago recognized in the theory of second order phase transitions. Reformulating the diagrammatic series for the correlation functions of the order parameter in terms of the bare interaction vertex but with the dressed correlation function with its suitable scaling exponent, one can check that this exponent is reproduced in each order of the perturbation theory. Besides one immediately encounters logarithmic ultraviolet divergences which arise in each order of the perturbation expansion. The logarithmic corrections are summed up to generate power corrections strongly renormalizing the naive exponents.

Fortunately this phenomenon does not occur in the theory of turbulence. As was demonstrated by Belinicher and L'vov [2] in qL variables there are neither infrared nor ultraviolet divergences in the diagrammatic expansion for G and F, if (3), (5) are assumed. The analogous theorem can be proven for high order correlation functions of u. This property is the ground for the assertion that in the consistent theory the simultaneous correlation functions actually have naive K41 exponents. (see also [6]). Nevertheless ultraviolet logarithms immediately arise in the diagrams for correlation functions of powers of the velocity gradient $\nabla_\alpha v_\beta$. The simplest example of such a correlation function is the following irreducible correlation function

$$K_{\epsilon\epsilon}(R) \equiv \langle\langle \varepsilon(t, \mathbf{r}) \varepsilon(t, \mathbf{r} + \mathbf{R}) \rangle\rangle \ . \tag{6}$$

Let us analyze the diagrammatic series for $K_{\epsilon\epsilon}(\mathbf{r}_1, \mathbf{r}_2)$. The first one-loop diagram for $K_{\epsilon\epsilon}(\mathbf{r}_1, \mathbf{r}_2)$ gives the expression possessing normal K41 behavior $\propto R^{-8/3}$. The diagrams of the next order are depicted in Fig.1, where circles designate the points $\mathbf{r}_1$ and $\mathbf{r}_2$, a vertex is determined by the nonlinear term in the

Navier-Stokes equation, a wavy line corresponds to the pair correlation function (3) and a combined wavy-straight line designates the Green's function (1). Using the estimates (3,5) we find that these diagrams give us the expressions for $K_{\epsilon\epsilon}$ which are $\propto R^{-8/3}$. However there are also logarithmic divergences in these diagrams related to the loops marked by the letter "r". Therefore the final answer behaves as $R^{-8/3}\ln(R/\eta)$. Generalizing the above analysis we conclude that diagrams of the n-th order will produce the normal K41 factor $R^{-8/3}$ with prefactors which are different powers of the logarithm up to the n-th power. Thus we encounter a series over the large logarithm $\ln(R/\eta)$, which could be an arbitrary function. Below we will argue that this function is an exponential one, that is a power of R/η. Such a function in the prefactor produces an anomalous scaling.

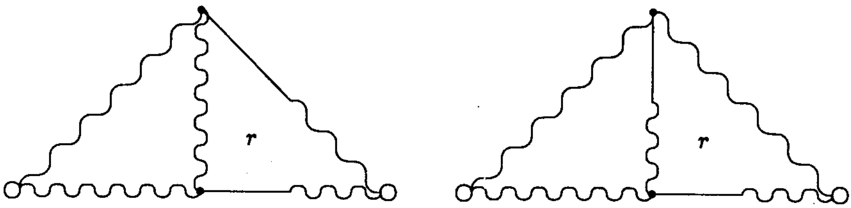

Fig.1. The first diagrams for $K_{\epsilon\epsilon}$ producing ultraviolet logarithms

In the framework of the Wyld technique a formally exact diagram representation for $K_{\epsilon\epsilon}$ can be formulated originating from the fact that in each diagram for $K_{\epsilon\epsilon}$ there exists only one cut going along all F-functions [3]. This enables us to formulate the representation depicted in Fig.2. There we have classified diagrams for $K_{\epsilon\epsilon}$ in accordance with the number of F-functions in our marked cut, the ovals designate objects which are sums of the blocks at the left and at the right sides of the marked cut.

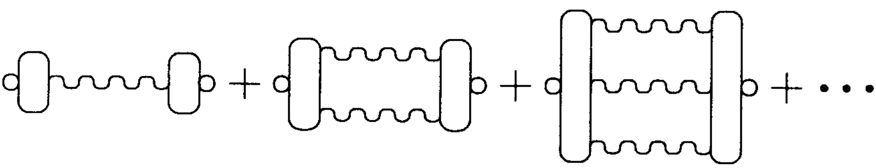

Fig.2. The formally exact diagrammatic representation for $K_{\epsilon\epsilon}$, the first terms of an infinite series

The first "one-bridge" term of the diagrammatic series depicted in Fig. 2 is actually reduced to the objects arising in the second "two-bridge" term. Therefore we begin our analysis with this three-leg object Υ corresponding to an oval in this "two-bridge" term. Designating by boxes the sums of the four-leg parts of the diagrams which cannot be cut along two lines, we come after summation to the diagrammatic relation presented in Fig.3, where the last term designates the bare contribution. The diagrammatic relation can be rewritten in analytical form, this gives the integral equation for Υ. The kernel B of this equation corresponds to the sum of boxes in Fig.3 with attached lines. Following the analysis given in [2, 3] one can demonstrate that there are neither ultraviolet nor infrared divergences in the higher order diagrams for B. This means that first contributions give the correct scaling for B. Utilizing (3,5) we conclude that the integration in the equation is dimensionless. It follows that this equation admits scaling solutions for Υ. Actually an infinite number of terms with different exponents present in Υ since the equation for Υ is an integral one. Analogously the higher-order terms of the series depicted in Fig.3 can be analized. Thus we conclude that $K_{\epsilon\epsilon}$ possess a complicated scaling behavior. The same is true for all correlation functions of of local fields $\varphi_j(\mathbf{r})$ constructed as different single-point products of the velocity gradients, since the gradients will produce logarithms and consequently anomalous dimensions.

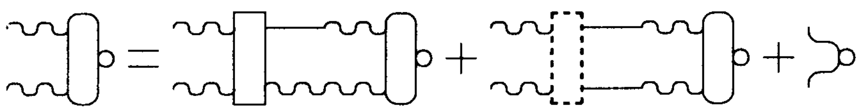

Fig.3. The diagrammatic equation for the three-leg object Υ entering the diagrammatic expression for $K_{\epsilon\epsilon}$

To proceed in the analysis of the scaling it is worthwhile to extract a set of local fields A_n with "clean" scaling behavior, namely each local field A_n is characterized by its scaling dimension Δ_n what means that

$$\langle A_n(\mathrm{R})A_m(0)\rangle \propto R^{-\Delta_n-\Delta_m}\,. \tag{7}$$

Among the set A_n one can extract the subset of the so-called primary fields O_n which give rise to all other fields A_n by differentiation. These "field-descendants" A_n are usually referred as *secondary fields.* The dimension Δ of any secondary field A differs from the dimension Δ_n of the corresponding primary field O_n by an integer number l: $\Delta=\Delta_n+l$, the number l being the number of the differentiations needed to obtain A from O_n. An example of a primary turbulent field is the velocity $\mathbf{v}$ itself possessing the normal K41 scaling dimension $\Delta_v=-1/3$.

Any local field φ_j can be expanded in a series over the fields A_n with some coefficients:

$$\varphi_j(\mathbf{r})=\sum_n \varphi_{j(n)}A_n(\mathbf{r})\,. \tag{8}$$

This expansion enables ones to reduce the correlation functions of φ_j to the correlation functions of the fields A_n. Unfortunately it is impossible to find the values of Δ_n but one can express the scaling behavior of observable quantities in terms of Δ_n. It is convenient to order the fields A_n over the values of their scaling dimensions: $\Delta_1 \le \Delta_2 \le \Delta_3 \ldots$. It is clear that the principal scaling behavior of correlation functions of φ_j is determined by the first nonzero term of its expansion into a series over A_n. For instance if the first terms of the expansions of φ_1 and φ_2 are not equal to zero the scaling behavior of the principal term in the correlation function $\langle\langle\varphi_1(\mathrm{R})\varphi_2(0)\rangle\rangle$ is $\propto R^{-2\Delta_1}$. We expect that this behavior is inherent to a correlation function of two scalar fields. Since (6) is just such function we conclude that $2\Delta_1=\mu$, where by definition $K_{\epsilon\epsilon}\propto R^{-\mu}$.

Now we can formulate for hydrodynamic turbulence fusion rules for fluctuating fields as introduced by Polyakov [7]. It is obvious that the product of the fields $A_n(\mathrm{r}_1)A_m(\mathrm{r}_2)$ taken at the nearby points behaves like a single-point object, which can be expanded into a series over $A_n(\mathrm{r})$. Thus we come to the relations

$$A_n(\mathrm{r}_1)A_m(\mathrm{r}_2)=\sum_l C_{mn,l}(\mathrm{r}_1-\mathrm{r}_2)A_l((\mathrm{r}_1+\mathrm{r}_2)/2)\,, \tag{9}$$

which are known as the operator algebra [8, 9]. The relations (8), (9) can be used in investigating any correlation function of the fields φ_j with two nearby points.

Special consideration is needed for the correlation functions of the first power of the velocity v and of its derivatives because of the incompressibility condition. For instance, the cross-correlation function of the velocity itself with any scalar field φ_j is equal to zero. To prove this, note that the correlation function $\langle\mathrm{v}(\mathrm{r})\varphi_j(0)\rangle$ is a vector which can only be directed along r. However we know that the divergence of this vector should be equal to zero because of incompressibility $\nabla\cdot\mathrm{v}=0$.

Note that if the system possesses conformal symmetry then there exists a set of strong selection rules for the coefficients in the r.h.s. of (7), established by Polyakov [10]. Namely these coefficients are not equal to zero for different values Δ_n and Δ_m only if these fields are secondary fields of the same primary field. This is the consequence of the "orthogonality rule": the correlation functions of different primary fields O_n are equal to zero if the system possesses conformal symmetry. This question arises in connection with the recent work of Polyakov [11] who treated $2d$ turbulence in the framework of the conformal approach (as is known [12] for $2d$ systems conformal symmetry permits one to establish many properties of the correlation functions, particularly possible sets of dimensions Δ_n, the conformal symmetry imposes also some restrictions on the r-dependence of correlation functions in $3d$ [13]).

Using (8), (9) one can examine the asymptotic behavior of correlation functions of the velocity differences. Consider the case when there are two sets of nearby points r_1, r_2 and r_3, r_4 separated by the large distance R. Take as an example the second power of the velocity difference $(\mathrm{v}_1-\mathrm{v}_2)^2$. Using (9) we can "fuse" this object into a single point. It is natural to expect that the principal term in this expansion is determined by O_1:

$$(\mathrm{v}_1-\mathrm{v}_2)^2\to f(r_{12})O_1((\mathrm{r}_1+\mathrm{r}_2)/2)\,, \tag{10}$$

where $r_{12}=|\mathrm{r}_1-\mathrm{r}_2|$. It means that

$$\langle\langle(\mathrm{v}_1-\mathrm{v}_2)^2(\mathrm{v}_3-\mathrm{v}_4)^2\rangle\rangle\propto f(r_{12})f(r_{34})R^{-2\Delta_1}\,, \tag{11}$$

remind that $2\Delta_1=\mu$. The r-dependence of the function $f(r)$ can also be established if one remembers that the general scaling behavior of the correlation function of velocity differences in the l.h.s. of (11) is determined by the conventional K41 index $-4/3$. Comparing this index with the scaling behavior (11) we conclude that $f(r)\propto r^{\Delta_1+2/3}$. Then we should take into consideration the terms of the expansion of $(\mathrm{v}_1-\mathrm{v}_2)^2$ over vector and tensor fields. These terms describe the dependence of $\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^2(\mathbf{v}_3-\mathbf{v}_4)^2\rangle\rangle$ on the angles between $\mathbf{R}$ and $\mathbf{r}_1-\mathbf{r}_2$, $\mathbf{r}_3-\mathbf{r}_4$. The main such term will be determined by the smallest value $\Delta_{;1}$ of principal scaling exponents of tensor fields.

The proposed scheme can easily be generalized for all even powers $(\mathbf{v}_1-\mathbf{v}_2)^{2n}$. The main contributions to the correlation function $\langle\langle(\mathbf{v}_1-\mathbf{v}_2)^m(\mathbf{v}_3-\mathbf{v}_4)^n\rangle\rangle$ are as follows: The first contribution $\propto R^{-2\Delta_1}$ will not depend on the angles, the second contribution $\propto R^{-\Delta_1-\Delta_{;1}}$ is the sum of two terms depending on the angle between $\mathbf{R}$ and $\mathbf{r}_1-\mathbf{r}_2$ or on the angle between $\mathbf{R}$ and $\mathbf{r}_3-\mathbf{r}_4$ only and the third term $\propto R^{-2\Delta_{;1}}$ depends on both angles. This is also the point where the conformal symmetry would reveal, since it kills the second contribution $\propto R^{-\Delta_1-\Delta_{;1}}$.

Let us now analyze the correlation functions of odd powers. First we consider the special case of the first power since the difference $\mathbf{v}_1-\mathbf{v}_2$ possesses the normal K41 dimension. The main term of the expansion of this difference in the series over local fields is $\nabla_\alpha v_\beta$. This means, e.g., that $\langle(v_{1\alpha}-v_{2\alpha})(v_{3\beta}-v_{4\beta})\rangle \propto R^{-4/3}$. Consider now the correlation function $\langle(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_3-\mathbf{v}_4)^{2n}\rangle$. As we have seen the correlation function $\langle \mathbf{v}O_n\rangle$ is zero for any scalar field O_n. Therefore only the vector and tensor fields A_n should be taken into account in the expansion for $(\mathbf{v}_3-\mathbf{v}_4)^{2n}$ what gives

$$\langle(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_3-\mathbf{v}_4)^{2n}\rangle \propto R^{-2/3-\Delta_{;1}}, \qquad (12)$$

where $\Delta_{;1}$ as above is the smallest exponent of tensor fields. Of course among the fields A_n in the expansion (8) for $(\mathbf{v}_3-\mathbf{v}_4)^{2n}$ there is a term with $\nabla_\alpha v_\beta$. This means that in any case there is the term $\propto R^{-4/3}$ in the correlation function $\langle(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_3-\mathbf{v}_4)^{2n}\rangle$. This is again the point where conformal symmetry could be checked: it admits only the behavior $\propto R^{-4/3}$.

Now consider a general odd power of the velocity difference $(v_{1\alpha}-v_{2\alpha})(\mathbf{v}_1-\mathbf{v}_2)^{2n}$. It can be expanded into a series over the same fields A_n as the even powers but with more complicated angular dependence of coefficients. Therefore the scaling behavior of the mutual correlation functions of the odd-odd and of the odd-even correlation functions at large separations will be the same as the behavior of the even-even correlation function. Terms with different scaling exponents can in principle be separated on the basis of their angular dependence.

The conclusions made in this article concern principal scaling behavior of correlation functions of velocity gradients and velocity differences. Therefore we hope that our predictions permit direct experimental checking.

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