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Dust-Alfvén Mach cones in Saturn's dense rings

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The formation of Mach cones associated with dust-Alfvén waves in Saturn's dense rings has been theoretically investigated. It is explicitly shown that for typical dusty plasma parameters corresponding to Saturn's dense rings, Mach cones are only formed by dust-Alfvén waves which are found to be more prominent than any other longitudinal waves (e.g. dust-acoustic waves). The characteristics of the dust-Alfvén Mach cones that are found to be formed in Saturn's dense rings are also presented. The dusty plasma model, dust-Alfvén waves, and dust-Alfvén Mach cones that we predict in our present letter are expected to be observed in Saturn's dense rings by the imaging and occultation experiments on board the NASA/ESA space mission CASSINI, arriving at Saturn in 2004.

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It is well known that an object moving with a supersonic speed in a dispersive medium creates a pressure disturbance that is not felt upstream from the object. The cone that confines the disturbance is called a Mach cone. The latter is well known in gas dynamics. They are produced, for example, by bullets and supersonic jet planes. If the perturbing object moves straight at a constant velocity U, it creates expanding waves that are circular in two-dimensions and spherical in three dimensions. The superposition of these waves forms a cone. The Mach cone opening angle θ, defined as a semivertex angle of the cone, is determined by the geometry as $\theta = \sin^{-1}(1/M)$ where $M = U/C_s$ is the Mach number of the supersonic object and C_s is the acoustic (sound) speed in the undisturbed medium.

Mach cones are also known to occur in gas dynamics [1, 2], solid matter [3], and in some crystals [4, 5]. In an elastic medium surrounding a fluid-filled borehole, spontaneously launched surface waves propagating along the fluid-solid boundary excite P and S waves propagating into the bulk solid. The interference between P and S waves forms Mach cones. The wavefront of the surface wave acts as the supersonic object as its speed is typically higher than the P and S waves.

Ship waves have an appearance similar to Mach cones. The latter are also known as the "Kelvin wedge" that forms behind a ship in deep water. Here a moving point-like disturbance generates either gravity or capillary waves on the fluid surface. These deep water strongly dispersive surface waves [6] are responsible for multiple Mach cone structures.

Besides the above mentioned Mach cones on human scales, Mach cones also occur in astronomical scales (e.g. the Earth's magnetotail formed by interaction with the solar wind) and microscopic scales (e.g. Cherenkov radiation created by rapidly moving elementary charge). Havnes et al. [7, 8] theoretically predicted the existence of super dust-acoustic Mach cones associated with dust-acoustic waves [9] of an unmagnetized dusty plasma, which are claimed to be relevant to Saturn's rings. Dubin [10] developed a linear theory for the phonon wake produced by a charge moving relative to a crystalline lattice in an unmagnetized plasma containing strongly coupled dust grains. The theory predicts multiple Mach cones due to constructive interference of strongly dispersive compressional phonons. However, Dubin's theory cannot be applied to Saturn's magnetized plasmas with weakly correlated dust grains. In Saturn's rings, we may have Mach cones associated with numerous dispersive plasma waves that are affected by the ambient magnetic field. For example, we may have obliquely propagating intermediate frequency ($\omega_{cd} \ll \omega \ll \omega_{ci}$, $k_z v_{te}$, where ω_{cd} and ω_{ci} are the dust and ion gyrofrequencies, respectively, k_z is the component of the wave vector $\mathbf{k}$ along the external magnetic field $\hat{\mathbf{z}}B_0$, and v_{te} is the electron thermal speed) long wavelength (in comparison with the electron Debye radius λ_{De} and the ion gyroradius ρ_s at the electron temperature) dust-acoustic waves [11] whose phase speed for $k_z\lambda_{De} \ll \ll \omega_{pd}/\omega_{pi}$ is $C_{De}/(1 + k_\perp^2\rho_s^2)^{1/2}$, where $C_{De} = \lambda_{De}\omega_{pd}$,

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$\rho_s = \lambda_{De}\omega_{pi}/\omega_{ci}$, ω_{pd} (ω_{pi}) is the dust (ion) plasma frequency, and $k_\perp$ is the component of $\mathbf{k}$ across $\hat{\mathbf{z}}$. Here, the electrons rapidly thermalize along the magnetic field direction and follow a Boltzmann distribution, while the density distributions of magnetized ions (unmagnetized dust grains) are affected (unaffected) by the external magnetic field. Furthermore, in the opposite limit, viz. $k_z\lambda_{De} \gg \omega_{pd}/\omega_{pi}$, the dust grains can be considered stationary, and the corresponding parallel (to $\hat{\mathbf{z}}$) phase speed of the dust ion-acoustic wave (DIAW) is $C_I = \lambda_{De}\omega_{pi}/(1 + k_\perp^2\rho_s^2)^{1/2}$, indicating that the DIA waves are non-dispersive along the magnetic field direction. The high-phase speed DIA waves would not participate in the Mach cone formation, as they propagate much faster than a dust boulder whose speed is given by Eq. (3). On the other hand, in the short wavelength (in comparison with the ion gyroradius ρ_i) dust acoustic fields, the ions follow a straightline orbit across $\hat{\mathbf{z}}$, and establish a Boltzmann density distribution in the wave potential. Taking a Boltzmann electron density distribution and inertial dust, one then obtains the dust acoustic wave (DAW) whose phase speed is $C_D = \lambda_D\omega_{pd}/(1+k^2\lambda_D)^2$, where $\lambda_D = \lambda_{De}\lambda_{Di}/(\lambda_{De}^2 + \lambda_{Di}^2)^{1/2}$ and λ_{Di} is the ion Debye radius. As usual, for $n_iT_e \gg n_eT_i$, the dust-acoustic phase speed is [11, 12] $C_d = (Z_d^2 n_d T_i/n_i m_d)^{1/2}$, where $n_i(n_d)$ is the ion (dust) number density, $T_e(T_i)$ the electron (ion) temperature, and Z_d is the number of electrons residing onto the dust grain surface. For typical plasma parameters corresponding to Saturn's rings [11,13–16], viz. $B_0 \simeq 0.2$ G, $T_i = 10$ eV, $n_d \simeq 10$ cm^{-3}, $Z_d \simeq 10^3$, $r_d \simeq 0.25$ μm, where r_d is the dust particle radius, one finds that $C_d^2/V_A^2 \simeq 5 \cdot 10^{-5}$, where $V_A = B_0/\sqrt{4\pi\rho_d}$ is the dust-Alfvén speed, $\rho_d = n_d m_d$ is the dust mass density, and m_d is the mass of micron-sized dust particles. This means that in Saturn's rings, the dust-magnetoacoustic wave propagation [17] is more prominent than the dust-acoustic wave propagation, and the waves involving perturbation of magnetic fields are likely to participate in the formation of Mach cones. Accordingly, in this Letter we have taken into account the dynamics of both the ion and dust species in the ambient Saturn's magnetic field, and study the dust-hydromagnetic waves and associated Mach cones in Saturn's dense rings. We have predicted here that the perturbation/acoustic waves that may exist in Saturn's rings are not the dust-acoustic, but are the dust-magnetoacoustic in which the magnetic pressure $B_0^2/4\pi$ gives rise to the restoring force and the dust mass density $n_d m_d$ provides the inertia. Therefore, in Saturn's dense rings, if Mach cones are formed, they are formed by the dust-magnetoacoustic waves, but, of course, not by the long and short wavelength dust-acoustic waves.

We consider a negatively charged dust particle of mass m_d and charge $-Z_d e$ moving in a field which includes Keplerian gravity, corotating planetary magnetic field (taken to be aligned centered dipole) with concomitant induced electric field [13–16, 18]. We first consider single particle dynamics and neglect the radiation pressure, plasma drag, planetary oblateness, charge fluctuations, and collective effects. The dynamics of such a negatively charged dust particle is governed by the combined gravitational, magnetic, and electric forces. The orbital angular velocity ω_d of the negatively charged dust particle can, therefore, be expressed as [13, 16]

$$\omega_d = \frac{1}{2r^3}\left[-\omega_{cd} \pm \sqrt{\omega_{cd}^2 + 4r^3(\Omega_k^2 + \omega_{cd}\Omega_p)}\right], \quad (1)$$

where r is the dust particle position normalized by the planet radius R_p, $\omega_{cd} = Z_d e B_0/m_d c$ and $\Omega_k = (GM_p/R_p^3)^{1/2}$ are the dust cyclotron and Kepler frequencies both evaluated at a point on the planetary equator, Ω_p is planetary spin rate, M_p is the planet mass, G is the universal gravitational constant, and c is the speed of light in vacuum. We note that in deriving (1) the planetary magnetic field $\mathbf{B_p}$ is assumed to be dipolar with the dipole strength $\mathbf{M} = \mathbf{B_0}R_p^3$ (where $\mathbf{B_0}$ is the magnetic field strength on the planetary equator), which is appropriate for Saturn and Jupiter. The + (−) sign in (1) represents the prograde (retrograde) motion of the dust particle.

A large boulder and a small dust particle will, therefore, move at difference velocities. The different in velocities V_d is given by

$$V_d = rR_p(\omega_d - r^{-3/2}\Omega_k). \quad (2)$$

To approximate V_d, let us consider the dust particle in Saturn's rings where [7, 13, 14, 16, 18] $R_p = 60300$ km, $M_p = 5.688 \cdot 10^{26} kg$, $\Omega_p = 1.691 \cdot 10^{-4}$ rad/s, $r = 7$, $B_0 \simeq 0.2$ G, $Z_d \simeq 10^3$, $r_d \simeq 0.25$ μm, so that $\Omega_k = 4.16 \cdot 10^{-4}$ rad/s and $\omega_{cd} \simeq 4.89 \cdot 10^{-5}$ rad/s. So for a particle in Saturn's rings we can safely take the approximations $\omega_{cd} \ll \Omega_k$, $\Omega_p \le \Omega_k$, and $r \ge 1$, which allow us to approximate (2) as

$$V_d \simeq \frac{R_p\omega_{cd}}{2r^2}\left(\frac{\Omega_p}{\Omega_k} - 1\right). \quad (3)$$

To study the perturbation of the medium, we consider a two-component magnetized dusty plasma composed of negatively charged dust grains and positively charged ions. We assume that the electron number density is highly depleted due to the attachment of almost all the electrons onto the surface of highly charged and extremely massive dust grains. The dust-ion plasma system is assumed to be immersed in a homogeneous magnetic field $\hat{\mathbf{z}}B_0$. This model is quite appropriate for Saturn's rings (e. g. Saturn's F-ring [11, 13 – 15, 19). We consider a small amplitude perturbation in such a dust-ion plasma, which may be described by two linearized coupled equations

$$\frac{\partial^2 \mathbf{u}_d}{\partial t^2} + \frac{V_A^2}{\omega_{ci}} \nabla \times \frac{\partial^2 \mathbf{B}}{\partial t^2} - V_A^2 \left(\nabla \times \frac{\partial \mathbf{B}}{\partial t} \right) \times \hat{\mathbf{z}} + C_d^2 \nabla (\nabla \cdot \mathbf{u}_d)] = 0, \quad (4)$$

and

$$\frac{\partial \mathbf{B}}{\partial t} + \hat{\mathbf{z}}(\nabla \cdot \mathbf{u}_d) - (\hat{\mathbf{z}} \cdot \nabla)\mathbf{u}_d - \frac{1}{\omega_{cd}} \nabla \times \frac{\partial \mathbf{u}_d}{\partial t} = 0, \quad (5)$$

where $\mathbf{u}_d$ is the dust fluid velocity, $\mathbf{B}$ is the wave magnetic field normalized to B_0, $\omega_{ci} = eB_0/m_i c$ is the ion gyrofrequency, and m_i is the ion mass. We now assume that the perturbation mode propagates in the x–z plane, i.e. $\mathbf{B}$ and $\mathbf{u}_d$ are proportional to $\exp[-i\omega t + i(k_x x + k_z z)]$. Therefore, using (4) and (5) we obtain

$$\begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = 0, \quad (6)$$

where

$$\begin{aligned} D_{xx,zz} &= (1 + k_{z,x}^2 \lambda_i^2)\omega - \frac{k_{z,x}^2 V_A^2 (\omega^2 - k_z^2 C_d^2)}{\omega(\omega^2 - k^2 C_d^2)}, \\ D_{xy} &= -D_{yx} = -ik_z^2 V_A^2 \left(\frac{1}{\omega_{cd}} - \frac{1}{\omega_{ci}} \right), \\ D_{xz} &= D_{zx} = \omega k_z k_x \lambda_i^2 + \frac{k_z k_x V_A^2 (\omega^2 - k_z^2 C_d^2)}{\omega(\omega^2 - k^2 C_d^2)}, \\ D_{yy} &= (1 + k^2 \lambda_i^2)\omega - \frac{k_z^2 V_A^2}{\omega}, \\ D_{yz} &= -D_{zy} = -ik_z K_x V_A^2 \left(\frac{1}{\omega_{cd}} - \frac{1}{\omega_{ci}} \right). \end{aligned} \quad (7)$$

Here, $\lambda_i = c/\omega_{pi}$ is the ion skin depth. We note that the origin of the dispersive effect involving the $k\lambda_i$ term is attributed to the ion inertial effect, which breaks the frozen-in-field condition.

We are interested in extremely low-frequency obliquely propagating dust-hydromagnetic waves for which $\omega \ll \omega_{ci}$ are valid. Using these approximations we obtain from (6) three types of obliquely propagating dust-hydromagnetic waves. These are

$$\frac{\omega}{k} = \frac{V_A \cos \delta}{\sqrt{1 + k^2 \lambda_i^2}}, \quad (8)$$

and

$$\frac{\omega}{k} = \frac{1}{\sqrt{2}} \left[\frac{V_A^2}{1 + k^2 \lambda_i^2} + C_d^2 \pm \pm \sqrt{ \left(\frac{V_A^2}{1 + k^2 \lambda_i^2} + C_d^2 \right)^2 - \frac{4 V_A^2 C_d^2 \cos^2 \delta}{1 + k^2 \lambda_i^2} } \right]^{1/2}, \quad (9)$$

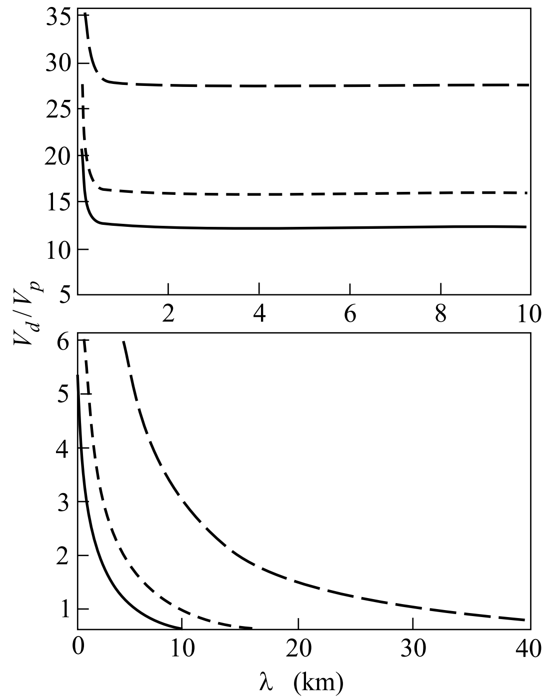
where δ is the angle between the directions of the ambient magnetic field and the wave propagation. Equation (8) represents the dispersion relation for the shear dust-Alfvén waves modified by the effect of the ion-skin depth which decreases their phase speed by the factor $\sqrt{1 + k^2 \lambda_i^2}$. On the other hand, (9) with + (−) sign represents the dispersion relation for the fast (slow) dust-hydromagnetic waves modified by the effect of the ion-skin depth and obliqueness. For typical plasma parameters corresponding to Saturn rings [11, 13 – 16], we find that $C_d^2/V_A^2 \simeq 5 \cdot 10^{-5}$. This means that in Saturn's rings, the dust-Alfvén wave propagation is much more prominent than the dust-acoustic wave propagation, and therefore the consideration of the dust-acoustic wave propagation and associated Mach cones [7, 8] in Saturn's rings are not realistic. We are, therefore, interested in examining the formation as well detecting the characteristics of Mach cones associated with the dust-hydromagnetic waves defined by (8) and (9).

As we have physically explained in the introduction, the Mach cones associated with the waves defined by (8) and (9) will be formed if the dust particle speed V_d is larger than the wave phase speed $V_p = \omega/k$, i.e. $V_d/V_p >> 1$. If this condition is satisfied, the Mach cone opening angle θ is given by

$$\theta = \sin^{-1} \left(\frac{V_p}{V_d} \right), \quad (10)$$

where V_d is given by (2) and $V_p = \omega/k$ is given by (8) for shear dust-Alfvén waves, and by (9) for fast and slow dust-magnetosonic waves. We have numerically analyzed V_d/V_p for typical plasma parameters corresponding to Saturn's rings [11, 13 – 16], given in the figure caption, and have shown that Mach cones are formed by the shear dust-Alfvén waves of wavelength ~ 7 km or less (cf. lower plot) and by slow dust-magnetosonic waves of any wavelength [without the upper bound on the wavelength (cf. upper plot)]. We have also found that the upper bound on the wavelength of both the shear dust-Alfvén and slow dust-magnetosonic waves by which Mach cones are formed increases as we increase their propagation angle δ (cf. lower and upper plots). We note here that for parameters corresponding to Saturn's rings Mach cones are not formed by the fast dust-magnetosonic waves.

It is obvious from figure and eq. (9) that the Mach cone opening angle θ decreases with increasing the prop-



Showing the wavelength regimes of shear dust-Alfvén (lower plot) and of slow dust-magnetosonic (upper plot) waves propagating with different angle δ for which Mach cones are formed in Saturn's dense rings ($B_0 = 0.2$ G, $T_i = 10$ eV, $n_d = 10\,\text{cm}^{-3}$, $Z_d = 10^3$, $r = 7$, $r_d = 0.25\,\mu$m). The solid, dotted and dash curves are for $\delta = 85$, $\delta = 87°$, and $\delta = 89°$ in both the plots

agation angle δ in the case of the shear dust-Alfvén waves. We have also estimated the Mach cone opening angle θ (that may be formed in Saturn's dense rings) associated with shear dust-Alfvén and slow dust-magnetosonic waves of wavelength of 5 km, propagating with an angle $\delta = 85°$. These are $\sim 30°$ and $\sim 4°$, respectively. Physically, Mach cones arise due to the constructive interference of dispersive dust-hydromagnetic waves in dust-ion plasmas of Saturn's dense rings. We expect that the NASA/ESA space probe CASSINI can make direct observations of the dust-hydromagnetic modes and associated Mach cones that we have reported in this Letter. From the opening angle of the dust-Alfvénic Mach cones, one can then deduce the dust and ion mass densities as well as dust charge and the optical depth of Saturn's dense rings. Although the present investigation provides the parameter regimes for the existence of the dust-Alfvénic Mach cones, the fine structure of a dust magnetoacoustic wake behind a dust boulder can be obtained by solving numerically the dust magnetoacoustic wave equation with an appropriate source. This investigation can be carried out on the lines of Brattli et al. [8].

In closing, we mention that although the formation of Mach cones in a magnetized dusty plasma of Saturn rings is attributed to constructive interference between linear dispersive dust magnetohydrodynamic waves, nonlinear and dissipative effects can appear when the wave amplitudes are large and dust charge perturbations are taken into consideration. A delicate balance between nonlinearity and dissipation can produce magnetoacoustic shock waves, similar to those studied by Popel et al. [20, 21] for an unmagnetized dusty plasma. However, a detailed investigation of the dust grain charging in a magnetized dusty plasma containing large amplitude dust hydromagnetic waves is quite involved, and is beyond the scope of the present work. We anticipate that dust charge perturbation effects might be insignificant since the dust grain charging time is typically much shorter than the timescale on which the dust magnetoacoustic perturbations develop.

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