

NON-EQUILIBRIUM NOISE IN A MESOSCOPIC CONDUCTOR: MICROSCOPIC ANALYSIS

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Current fluctuations are studied in a mesoscopic conductor using non-equilibrium Keldysh technique. We derive a general expression for the fluctuations in the presence of a time dependent voltage, valid for arbitrary relation between voltage and temperature. Two limits are then treated: a pulse of voltage and a DC voltage. A pulse of voltage causes phase sensitive current fluctuations for which we derive microscopically an expression periodic in $\int V(t)dt$ with the period h/e . Applied to current fluctuations in Josephson circuits caused by phase slips, it gives an anomalous contribution to the noise with a logarithmic singularity near the critical current. In the DC case, we get quantum to classical shot noise reduction factor $1/3$, in agreement with recent results of Beenakker and Büttiker.

Introduction. Recently, the Landauer approach to electric transport [1] was extended to describe current fluctuations [2, 3, 4]. The central result, formulated for a single channel conductor with transmission coefficient T , is that at zero temperature the current noise magnitude is given by $(e^2/h)T(1-T)eV$, where V is the drop of voltage across the system. Thus the quantum noise comes to be a factor $1-T$ below the classical shot noise level eI . Beenakker and Büttiker [5] generalized this picture to a mesoscopic conductor, where there are many conducting channels with a distribution of transmission constants T_n . The noise is a sum of contributions of separate channels: $\sum_n T_n(1-T_n)e^3V/h$. Since in the limit of large dimensionless conductance G the distribution of T_n is provided by the random matrix theory in the universal form, $P(T)dT = Gdx$, where $T = 1/\cosh^2 x$, one obtains for the noise $\frac{1}{3}Ge^3V/h$, i.e. the theory predicts universal quantum to classical noise ratio $1/3$. Another approach developed recently by Nagaev, who used kinetic equation[6] and with this technique also arrived at the factor $1/3$.

In order to understand better the relation of these results with the conventional many-body methods, it would be of interest to do a microscopic calculation. In this paper, we study noise using non-equilibrium Keldysh technique and derive a more general formula for current fluctuations caused by a time dependent voltage. In the DC limit our results agree with those found by other methods, and give a generalization to an AC situation. We study current-current correlation $\langle\langle I(t_1)I(t_2) \rangle\rangle$ in a mesoscopic conductor under a slowly varying voltage and show that it is a periodic function of $\Phi_{12} = c \int_{t_1}^{t_2} V(t)dt$ with the period $\Phi_0 = hc/e$, the single electron flux quantum. With this, we treat current fluctuations in a metal caused by a short pulse of voltage, or, equivalently, by a varying magnetic flux. In this case the current fluctuations show phase sensitivity (nonstationary Aharonov-Bohm effect) lacking in the DC case [7]. Recently, for a single channel conductor geometry, the phase sensitive noise was expressed through the transmission coefficient T [8]. We derive it below for a mesoscopic conductor within Keldysh formalism, and show

that the magnitude is given by the normal metal conductance G reduced by the same factor $1/3$.

We do the calculation for a cylindrical contact between two ideal leads. As a simplest mesoscopic point of view, we assume diffusive regime inside the contact and completely ignore phase breaking effects, except for the temperature. Also, we treat electrons as noninteracting fermions. This somewhat restricts the range of applicability of our results, since in real systems interactions are important. However, even for interacting fermions, our calculation remains valid as long as the Fermi liquid picture holds.

Main results. For the calculation we take two arbitrary sections x_1, x_2 of the contact, and find the current-current correlator $S_{x_1, x_2}(t_1, t_2) = \langle \hat{I}_{x_1}(t_1) \hat{I}_{x_2}(t_2) \rangle$. It will be explicit in the calculation that $S_{x_1, x_2}(t_1, t_2)$ does not depend on the choice of the sections x_1, x_2 , when the time scale τ^* on which the voltage $V(t)$ varies is much longer than the time of diffusion through the contact. Under the assumption of large τ^* , for a cylindrical contact our result reads

$$S(t_1, t_2) = S_{eq}(t_1 - t_2) \left(1 - \frac{1}{6} |1 - e^{i\phi_{12}}|^2 \right), \quad \phi_{12} = \frac{2\pi c}{\Phi_0} \int_{t_1}^{t_2} V(t) dt, \quad (1)$$

where $S_{eq}(t_1 - t_2)$ is the correlation in equilibrium:

$$S_{eq}(t) = \int e^2 G \omega \coth \frac{\omega}{2T} e^{-i\omega t} \frac{d\omega}{4\pi^2} = -\text{Re} \frac{e^2 G T^2}{2 \sinh^2 \pi T(t + i0)}. \quad (2)$$

The correlator exhibits Fermi anticorrelation in the time domain, since it is explicitly negative at any $t_1 \neq t_2$. Let us emphasize, however, that the corresponding fluctuation of transmitted charge is positive, due to a compensation with a singularity in $S_{eq}(t_1, t_2)$ at $t_1 = t_2$. Also, since the second term of Eq.(1) gives positive contribution to the correlator, the inequality holds: $S(t_1, t_2) \geq S_{eq}(t_1 - t_2)$, which means that the excess noise is strictly positive.

At constant applied voltage, $\phi_{12} = \frac{e}{\hbar} V(t_2 - t_1)$, the correlator $S(t_1, t_2)$ is a function only of $t_1 - t_2$, and thus it can be characterized by a spectral density:

$$S_\omega = \frac{1}{6} (4S_\omega^{eq} + S_{\omega+eV}^{eq} + S_{\omega-eV}^{eq}), \quad (3)$$

where $S_\omega^{eq} = e^2 G \omega \coth \frac{\omega}{2T}$ is the equilibrium Nyquist noise spectrum. Then, the noise at low frequency $\omega \ll T, eV$ is given by

$$S_0 = \frac{1}{3} e^2 G (4T + \coth \frac{eV}{2T}).$$

In the limit $T = 0$ this gives $S_0 = \frac{1}{3} e^2 G V$, the celebrated quantum shot noise result [5, 6].

Beyond the DC situation, Eq.(1) allows to study noise in any AC setup, e.g., current fluctuations due to varying magnetic flux or due to a pulse of voltage. Physically, in this case the system can be realized as a normal metal ring in an AC magnetic field, or as a shunt resistor of a superconducting circuit in the regime of the nonstationary Josephson effect. We consider a step-like time dependence of

the flux, corresponding to a pulse of voltage, and for $T = 0$ derive an expression for the fluctuations of transmitted charge:

$$\langle\langle \delta Q^2 \rangle\rangle = \frac{1}{3} e^2 G \left(\frac{\Phi}{\Phi_0} + \frac{2}{\pi^2} \sin^2 \pi \frac{\Phi}{\Phi_0} \ln \frac{t_0}{\tau^*} \right), \quad (4)$$

where $\delta Q = \int_{-t_0}^{t_0} \hat{I}(t') dt'$ is the charge transmitted over the interval $-t_0 < t < t_0$, Φ is the height of the flux step, and τ^* is the duration of the step, assumed to be much shorter than t_0 . A similar expression was derived recently for a single channel conductor [8]. In Eq.(4) the first term corresponds to the $\omega = T = 0$ noise (3) integrated over time, since the flux and the voltage are related, $V(t) = -\frac{1}{c} \dot{\Phi}(t)$. The second Φ_0 -periodic term with an infrared logarithmic divergence corresponds to the non-stationary AB effect [7]. We propose to observe it in a Josephson circuit with a shunt, where it causes an anomalous contribution to the noise in the shunt, diverging as I approaches I_c .

General formalism. Let us turn to the calculation. In the current-current correlator $\langle\langle j(r_1, t_1) j(r_2, t_2) \rangle\rangle$ we take the times t_1 and t_2 on different branches of the Keldysh contour, and write $\langle\langle j_1 j_2 \rangle\rangle$ through the functions G^{+-} and G^{-+} as $\langle\langle j_1 j_2 \rangle\rangle = \text{tr}(j_1 G_{12}^{+-} j_2 G_{21}^{-+})$. The trace tr means integration over inner momenta and energy, summation over spin indices and averaging over configurations of the random potential. After expressing Fourier transform of the correlator in terms of retarded and advanced Green's functions G^R and G^A , and the Keldysh function F , we get

$$\langle\langle j j \rangle\rangle_{k, \omega} = \frac{1}{4} \text{tr}(j_{k, \omega} F j_{-k, -\omega} F) + \frac{1}{2} \text{tr}(j_{k, \omega} G^R j_{-k, -\omega} G^A). \quad (5)$$

The functions $G^{R(A)}$ are familiar: $G_{\epsilon, p}^{R(A)} = (\epsilon - \xi_p \pm i/2\tau)^{-1}$. The function F satisfies Dyson's equation [9]. With the condition of short mean free path $l \ll L$, it is reduced to the diffusion equation for the quantity $\frac{i}{\tau} \bar{s}(\mathbf{r}, t_1, t_2)$ defined according to $F = G^R \bar{s} - \bar{s} G^A = \frac{i}{\tau} G^R \bar{s} G^A$. Following the usual scheme [10, 11], we treat the vertex \bar{s} as a two-time diffuson [11, 12] that satisfies the equation

$$(\partial_{t_+} - D \bar{\nabla}^2) \bar{s}(\mathbf{r}, t_1, t_2) = 0, \quad t_+ = \frac{1}{2}(t_1 + t_2), \quad (6)$$

together with the condition on the boundary with the leads :

$$\bar{s}(\mathbf{r}, t_1, t_2) = \frac{i}{\tau} \bar{s}_0(t_1 - t_2), \quad (7)$$

where $\bar{s}_0(t) = \int e^{-i\epsilon t} \tanh \frac{\epsilon}{2T} \frac{d\epsilon}{2\pi}$, i.e., the leads serve as reservoirs of equilibrium electrons. The vector potential $A(\mathbf{r}, t)$ enters Eq.(6) through $\bar{\nabla} = \nabla - i \frac{e}{c} A(\mathbf{r}, t_1) + i \frac{e}{c} A(\mathbf{r}, t_2)$. Initially, at $t_+ = -\infty$, $\bar{s}(\mathbf{r}, t_1, t_2) = \bar{s}_0(t_1 - t_2)$ everywhere in the contact. Note, that our definition of the diffuson differs from that of [12] because \bar{s} is the vertex of the function F and thus it is a solution of the kinetic equation, which in this case is reduced to Eq.(6) in a conventional way.

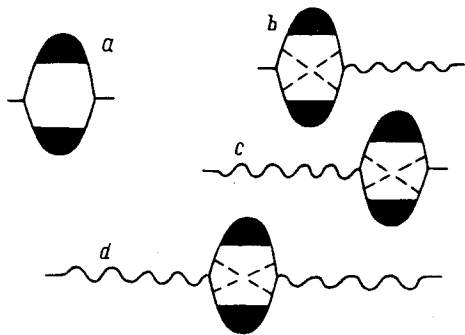
Before we discuss averaging over disorder, let us evaluate the simplest diagram shown in Fig.a. This contribution to the current correlation is local, since it decays at distances $\gg l$. Thus we write it through the conductivity σ as

$$\langle\langle \delta j^\alpha(r_1, t_1) \delta j^\beta(r_2, t_2) \rangle\rangle = -\frac{1}{2} e^2 \sigma \text{Re} [\bar{s}(\mathbf{r}, t_1 + i0, t_2) \bar{s}(\mathbf{r}, t_2 - i0, t_1)] \delta_{\alpha\beta} \delta(\mathbf{r}_1 - \mathbf{r}_2), \quad (8)$$

with the regularization at equal times that follows from the second term of Eq.(5). Now, we can dress any current vertex of Fig.a by a diffusion ladder, which gives the other three diagrams of Figure. Analytic elements are standard: the factor $-iDk\epsilon\pi\nu_0\tau$ corresponds to the current vertex and the factor $(\pi\nu_0\tau)^{-1}/(-i\omega\tau + Dk^2\tau)$ corresponds to the diffuson. In the limit $\tau^* \gg L^2/D$ one sets $\omega = 0$ in the diffusons, and then the sum of all contributions to the noise, the non-local (Fig.b,c,d) and the local (Fig.a), is given by

$$\langle\langle j^\alpha(t_1)j^\beta(t_2) \rangle\rangle_k = \left[\delta_{\alpha\alpha'} - \frac{k_\alpha k_{\alpha'}}{k^2} \right] \langle\langle \delta j^{\alpha'}(t_1) \delta j^{\beta'}(t_2) \rangle\rangle_k \left[\delta_{\beta\beta'} - \frac{k_\beta k_{\beta'}}{k^2} \right] \quad (9)$$

One verifies that, due to the projector form of the expressions in the brackets [...], the correlator of currents through two arbitrary sections is independent of the choice of the sections.



Graphs for current fluctuations are shown: (a) the local; (b,c,d) the non-local. Black vertices represent Keldysh function $\frac{1}{\tau}\bar{s}$ (see Eqs.(6),(7)), wavy lines are diffusons, and each box is dressed by impurity lines to form a Hikami vertex [13]

The result (9) allows a simple and natural interpretation in terms of a diffusion equation with random current source, $\frac{\partial n}{\partial t} = D\nabla^2 n - \nabla\delta j$. Given Eq.(8) for the fluctuations of δj , one gets an expression for the fluctuation of the total current $j = -D\nabla n + \delta j$ equal to the sum of the graphs of Figure, which in the low frequency limit is reduced to Eq.(9).

Cylindrical geometry. The rest of our discussion depends on the specific shape of the contact. Let us consider a cylindrical contact of length L between two ideal leads that serve as reservoirs of equilibrium electrons.

Let us solve Eq.(6) assuming that the time scale τ^* on which the field varies is longer than the diffusion time, $\tau^* \gg L^2/D$:

$$\bar{s}(x, t_1, t_2) = \bar{s}_0(t_2 - t_1) \left[1 - \frac{x}{L} + \frac{x}{L} e^{i\phi(t_2) - i\phi(t_1)} \right] e^{-i\phi(x, t_2) + i\phi(x, t_1)}, \quad (10)$$

where x is the coordinate along the cylinder axis,

$$\phi(x, t) = \frac{2\pi}{\Phi_0} \int_{-\infty}^x A(x', t) dx'$$

$0 < x < L$, and $\phi(t) = \phi(L, t)$. The solution obeys boundary conditions (7) at $x = 0$ and $x = L$. With $\tau^* \gg \frac{L^2}{D}$, one comes to Eq.(10) by neglecting the time derivative of $\bar{s}(r, t_1, t_2)$ in Eq.(6) with respect to the space derivative. Then the vector potential in Eq.(6) can be eliminated by a gauge transformation, where it should

not be forgotten that the gauge transformation changes the boundary conditions for the new function \bar{s}' :

$$\bar{s}'|_{x=L} = \bar{s}_0(t_2 - t_1)e^{i\phi(t_2) - i\phi(t_1)}.$$

Transformed function \bar{s}' satisfies the usual Laplace's equation which is readily solved.

In the cylindrical geometry the quantity k^{-2} in Eq.(9) should be interpreted as the Green's function $\mathcal{D}(\mathbf{r}_1, \mathbf{r}_2)$ of the Laplace's operator ∇^2 . According to Eq.(10), the source fluctuations (8) are homogeneous in every section of the cylinder, i.e., they depend only on x . Thus the problem becomes effectively one dimensional, and we can write :

$$\mathcal{D}(x_1, x_2)_{\omega=0} = \begin{cases} x_1(L - x_2)/L, & x_1 < x_2 \\ x_2(L - x_1)/L, & x_2 < x_1 \end{cases} \quad (11)$$

We substitute $\partial_{x_1}\mathcal{D}(x_1, x_2)\partial_{x_2}$ for $k^x k^x/k^2$ in Eq.(9), and readily get the result (1).

AC voltage noise. Now, we shall consider the fluctuations caused by a pulse of voltage of duration $\tau^* \ll \hbar/T$. By setting $T=0$ in Eq.(1) we get

$$S(t_1, t_2) = -\frac{1}{3} e^2 G \operatorname{Re} \frac{|1 - e^{i\phi_{12}}|^2}{4\pi^2(t_2 - t_1 + i0)^2} \quad (12)$$

Let us calculate the fluctuation of charge transmitted during the interval $-t_0 < t < t_0$:

$$\langle\langle \delta Q^2 \rangle\rangle = \int_{-t_0}^{t_0} \int_{-t_0}^{t_0} S(t_1, t_2) dt_1 dt_2.$$

The logarithmically diverging term of Eq.(4) is obtained from (12) by integrating over the times t_1 and t_2 before and after the voltage pulse ($t_0/\tau^* \rightarrow \infty$)

$$\langle\langle \delta Q^2 \rangle\rangle_{\log} = \frac{e^2 G}{3\pi^2} \sin^2 \frac{\pi\Phi}{\Phi_0} \left[\int_{t_0}^{-\tau^*} dt_1 \int_{\tau^*}^{t_0} \frac{dt_2}{(t_2 - t_1)^2} + (t_1 \leftrightarrow t_2) \right] = \frac{e^2 G}{3\pi^2} 2 \sin^2 \frac{\pi\Phi}{\Phi_0} \ln \frac{t_0}{\tau^*} \quad (13)$$

The contribution proportional to Φ/Φ_0 is extracted from almost coinciding times t_1 and t_2 . For that, we integrate (12) over $t = \frac{1}{2}(t_1 + t_2)$ and $t' = t_2 - t_1$. Assuming the time dependence of ϕ smooth and monotonous, we write $\phi_{12} = \dot{\phi}(t) t'$, do the integral over t' , and with $\dot{\phi}(t) > 0$ come to the first term of Eq.(4). At finite temperature, Eqs.(13),(12),(4) hold for $t_0 \leq \hbar/T$, otherwise t_0 has to be replaced by \hbar/T .

Let us consider a Josephson junction shunted by a normal metal resistor. If the current is fixed above the critical, $I > I_c$, the voltage on the resistor oscillates with the Josephson frequency $\omega = 2e\bar{V}/\hbar$:

$$V(t) = R \frac{I^2 - I_c^2}{I + I_c \cos \omega t}, \quad (14)$$

with $R^{-1} = \frac{e^2}{\hbar} G$ and $\bar{V} = R\sqrt{I^2 - I_c^2}$. At $I - I_c \ll I_c$, the signal (14) corresponds to a periodic sequence of steps in the flux, of the duration $\tau^* = RI_c$ and of the height

$\Phi = \frac{1}{2}\Phi_0$ each. Let us use Eq.(4) derived, for a single step to estimate the current noise in the shunt caused by the signal (14). The step height Φ corresponds to the half-period of Eq.(4), and thus we get a logarithmically diverging noise associated with each step. For the low frequency noise spectrum this gives

$$S_{\omega=0} = \frac{1}{3}e^2G\bar{V} \left\{ 1 + \frac{1}{2\pi^2} \ln \frac{RI_c}{\bar{V}} \right\}, \quad (15)$$

where the temperature is assumed to be small, $T \ll e\bar{V}$. The noise (15) is anomalously large, since it exceeds the shot noise level $\frac{1}{3}e^2G\bar{V}$ by a logarithmic factor, diverging as $I \rightarrow I_c$.

Experimentally, the observation of the noise (15) is more feasible in the temperature range $e\bar{V} < T < eRI_c$ [14], where it gives a correction to the thermal noise [15]. For such temperatures, $\ln RI_c/\bar{V}$ in Eq.(15) should be replaced by $\ln eRI_c/T$.

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