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HEAT TRANSFER FOR LAMINAR FLOW ACROSS PLATES AND CYLINDERS OF VARIOUS CROSS SECTION

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Local and over-all Nusselt numbers were calculated for a potential flow a across flat plate for the case of large or moderate Péclet number (Pe). The method of solution is based on multiple applications of the Wiener-Hopf method by solving the integral equation for the power source function. Conformal transforms provides Nusselt numbers for cylinders of various cross section. The accuracy of the method is of the order of $\exp(-2Pe)$.

The problem of heat transfer for a body moving through a liquid is a classical problem of the fluid mechanics (see review [1]). Boussinesq [2] first used the conformal transformation to reduce the problem for the cylinder in a potential stream of an inviscid liquid into the problem for a thin plate in a uniform stream. But this simple case also has no exact solution .

The model of an inviscid liquid it is possible to apply for the flow of liquids with very low Prandtl number (like liquid metals). Since the Prandtl number is the ratio of momentum to thermal diffusivity, the thermal-boundary-layer thickness will be much greater than that of the momentum layer. Low Prandtl numbers correspond to large Reynolds numbers (for large or moderate Péclet numbers). The stream is laminar for flows past slender bodies only.

The second example is the flow past bodies with a free surface (like a gas bubble [3]). If a laser beam is used as a power source (e. g. by welding), a long, almost cylindrical cavity (keyhole) is formed through the metal to be welded (see e.g. [4]). The temperature of the keyhole surface is equal to the evaporation temperature of the metal T_v. In the case of the electron beam welding electrons directly heat the metal and the keyhole cross section reproduces the shape of the electron beam (circular, elliptic or other cylinder). The surface of the keyhole is a free surface for liquid metal and the flow remains laminar at large Reynolds numbers for a bluff (e.g. circular) cylinder also [5]. The aim of the present paper is to obtain analytical formulas describing the heat transfer into the fluid at large and moderate Péclet numbers.

First we consider a thin plate with length L (in x-direction) in a stream with constant velocity $v_x = v_0$, $v_y = 0$. The thermoconductivity equation (in the restframe of the plate) is

$$\mathrm{v}\nabla T - \kappa\Delta T = 0 \ , \tag{1}$$

where κ is thermodiffusivity of the liquid. The border conditions correspond to a constant temperature $T = T_v$ at the plate and the ambient temperature T_0.

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We use dimensionless variables: $T = (T - T_v)/(T_v - T_0)$ and the unit of length is $l_0 = v_0/(2\kappa)$. The dimensionless length of the plate is known as the Péclet number (based on $L/2$): $a = Lv_0/(2\kappa) = Pe$.

The dimensionless thermoconductivity equation becomes

$$2\frac{\partial T}{\partial x} - \Delta T = 2h_p(x)\delta(y) \ . \tag{2}$$

Here $h_p(x)$ is the surface-heat-flux distribution (the local Nusselt number) for one side of the plate. Note, that $h_p(x)$ has nonzero value at the plate only. Our purpose is to find $h_p(x)$ that provides $T = 1$ at the plate ($0 < x < a$ and $y = 0$) and $T = 0$ at infinity.

The substitution

$$W(x,y) = T(x,y)e^{-x} \ , \quad \rho(x) = h_p(x)e^{-x} \tag{3}$$

transforms the thermoconductivity equation into the Helmholtz equation

$$W - \Delta W = 2\rho(x)\delta(y) \tag{4}$$

with the border conditions $W = e^{-x}$ at $0 < x < a$ and $y = 0$. The Green function of the Helmholtz operator is $K_0(r)/(2\pi)$ (see e.g [6]), where $K_0(r)$ it the modified Bessel function [7]. It provides an integral equation for $\rho(x)$

$$\int_0^a \frac{K_0(|\, x - s\,|)}{\pi}\rho(s)ds = e^{-x} \qquad \text{at} \qquad 0 < x < a \ . \tag{5}$$

Unfortunately, the right-side is a known function at the finite interval $(0, a)$ only and it is impossible to use directly the Fourier transformation to solve this equation. Now we use the Wiener-Hopf method (see e.g. [6]) to obtain the exact solution in the case of an infinite a and an approximate solution in the case of large finite a.

In the case of an infinite a the function $\rho(x) = \rho_0(x)$ is equal to zero at $x < 0$. Let us denote as $f^+(x) = e^{-x}$ the right side of integral equation (5) at $x > 0$. The value of the right side at $x < 0$ is an unknown function $g^-(x)$. Index '+' (or '−') shows that the function is not zero at positive (negative) x only. The convolution theorem provides an algebraic equation for the Fourier transforms of $\rho_0(k)$, $f^+(k)$ and $g^-(k)$ (all Fourier transforms appearing in the text are listed in [8]):

$$\frac{\rho_0(k)}{\sqrt{1+k^2}} = f^+(k) + g^-(k) \ . \tag{6}$$

Functions $\rho_0(k)$ and $f^+(k) = (1 - ik)^{-1}$ are analytical functions of complex k in the upper half-plane and $g^-(k)$ is an analytical function in the lower half-plane. In accordance with the Wiener-Hopf method we must collect all terms analytical in the upper (lower) half-plane in the left (right) side. It provides

$$\frac{\rho_0(k)}{\sqrt{1-ik}} - \frac{\sqrt{2}}{1-ik} = g^-(k)\sqrt{1+ik} + \frac{\sqrt{1+ik}-\sqrt{2}}{1-ik} \ . \tag{7}$$

To calculate $\rho_0(x)$ we must remain all terms analytical in the upper half-plane only. It provides

$$\rho_0(k) = \frac{\sqrt{2}}{\sqrt{1-ik}} \quad \text{and} \quad \rho_0(x) = e^{-x}\sqrt{\frac{2}{\pi x}} \ . \tag{8}$$

The next step is the solution for finite a. The function $\rho_0(x)$ satisfies to eq. (5) but it isn't equal to zero at $x > a$. However, we can improve the solution by multiple using the Wiener-Hopf method in the shifted coordinate system. In the following text the index '+' ('−') denotes the function that isn't equal to zero at $x > a$ ($x < a$) only. The function $\rho_0(x)$ is a sum of two functions: $\rho_0^+(x) = \rho_0(x)\Theta(x-a)$ and $\rho_0^-(x) = \rho_0(x) - \rho_0^+(x)$. If we cut $\rho_0(x)$ at $x > a$, we must add some function $\rho_1^-(x)$ to compensate the change of the left side in (5) at $x < a$. The change of the right side in eq. (5) at $x > a$ is an unknown function $g^+(x)$. Corresponding equation for the Fourier transforms is

$$\frac{\rho_1^-(k) - \rho_0^+(k)}{\sqrt{1+k^2}} = g^+(k) \tag{9}$$

or

$$\frac{\rho_1^-(k) + \rho_0^-(k)}{\sqrt{1+ik}} = \frac{\sqrt{2}}{2\sqrt{1+k^2}} + g^+(k)\sqrt{1-ik} \ . \tag{10}$$

The left side of the last equation is an analytical function in the lower half-plane, the last term in the right side is an analytical function in the upper half-plane. Now we must decompose the first term in the right side as the sum of two functions analytical in the upper or lower half-plane. This term is the Fourier transform of $\sqrt{2}K_0(|\,x\,|)/\pi$. If we set this function to be zero at $x > a$, the Fourier transform will be an analytical function in the lower half-plane. The Fourier transform of the rest will be an analytical function in the upper half-plane. Taking into account all terms analytical in the lower half-plane, we obtain

$$\rho(k) = \rho_1^-(k) + \rho_0^-(k) = \frac{\sqrt{2(1+ik)}}{\pi}\int_{-\infty}^{a} K_0(|\,x\,|)e^{ikx}dx \ . \tag{11}$$

The inverse Fourier transform provides $\rho(x)$ and the local Nusselt number (3):

$$h_p(x) = \sqrt{\frac{2}{\pi x}} + \frac{1}{\sqrt{2}\pi^{3/2}}\int_a^\infty \frac{e^{2x-s}}{(s-x)^{3/2}}K_0(s)ds \ . \tag{12}$$

The over-all Nusselt number is the mean value of $h_p(x)$ times characteristic length ($=a/2$) of the problem.

$$Nu_p(a) = \frac{1}{2}\int_0^a e^x \rho(x)dx = \frac{\rho(k=-i)}{2} \ . \tag{13}$$

The integral in (11) for $k = -i$ has an analytical expression [7]. It provides

$$Nu_p(a) = \frac{ae^a(K_0(a) + K_1(a))}{\pi} \ . \tag{14}$$

The plot of $Nu_p(a)$ is shown in Fig.1. Equations (12) and (14) are the main results of our calculations.

In the case of very large a we can use the asymptotic expansion for $K_n(s)$ at large s [7]. It provides

$$Nu_p(s) \approx \sqrt{\frac{2a}{\pi}} \ . \tag{15}$$

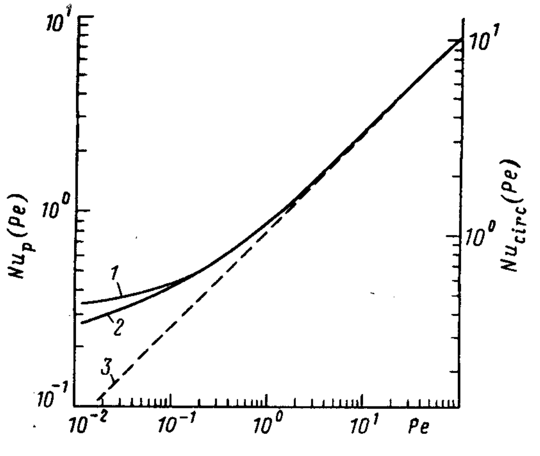

Fig.1. The over-all Nusselt number for a flat plate and circular cylinder as a function of the Péclet number: *1* – analytical expressions (14) and (18); *2* – numerical results from [9]; *3* – asymptotic expression for very large Pe (15)

Within this accuracy the local Nusselt number is

$$h_p(x) \approx \sqrt{\frac{2}{\pi x}} + \frac{e^{-2(a-x)}}{\pi\sqrt{a(a-x)}} + \sqrt{\frac{2}{a\pi}}\mathrm{erfc}(2\sqrt{a-x}) \ . \qquad (16)$$

The plot of $h_p(x)$ is shown in Fig.2.

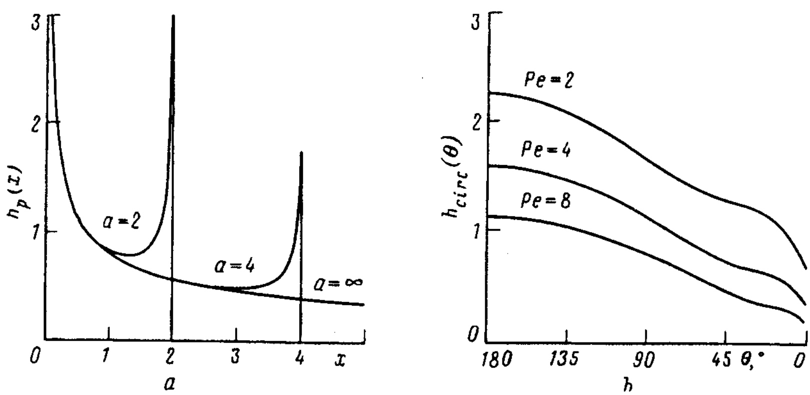

Fig.2. The local Nusselt number for a plate (a) and circular cylinder (b) for some values of the Péclet number

In contrast to $\rho_0(x)$, the function $\rho(x) = \rho_0^- + \rho_1^-(x)$ is equal to zero at $x > a$. But this function is not zero at $x < 0$. However, eq.(16) shows, that the contribution to the Nusselt number from the region $x < 0$ is of the order of $\exp(-2a)$. This is the accuracy of our method. We also can use the Wiener-Hopf method repeatedly: cut the function $\rho_1(x)$ at $x < 0$ and introduce a new function ρ_2 to compensate this cutting at $x > 0$ and so on. The idea is transparent, but each stage needs calculation of complicated sequences of the Fourier transformations or convolutions. Comparison with results of numerical calculations [9] (see Fig.1) shows, that eqs. (12)-(14) provide an accuracy of the order of 10^{-2} at $a = 0.3$ and the error rapidly (exponentially) vanishes with increasing a.

Nusselt numbers for various cylinders may be obtain by conformal transformation [2]. Let $z(w)$ be a conformal transformation collapsing the border of the cylinder

C at the complex plane $w = \zeta + i\eta$ into a line with length L at the plane $x = x + iy$ and $z'(w)$ tends to unity at large $\mid w \mid$. Function $z(w)$ is the (scaled) complex potential of the stream with velocity v_0 at infinity: $v_\zeta(w) - iv_\eta(w) = v_0 z'(w)$. The thermoconductivity equation is invariant relating to conformal transformation. The single parameter of the problem is the (dimensionless) length of the line: $a = Lv_0/(2\kappa) = Pe$ (the Péclet number). The local Nusselt number of the cylinder h_c is simply related to the solution for the plate h_p by:

$$h_c(w) = h_p(x) \mid z'(w) \mid \quad . \tag{17}$$

The over-all heat flux over the cylinder is a function of the Péclet number only. It provides the over-all Nusselt number. Some examples:

Circular cylinder with radius R_0, $w = Re^{i\theta}$. The complex transform is $z = w + R_0^2/w + 2R_0$ and $L = 4R_0$. It provides the Péclet number and the over-all Nusselt number (based on $2R_0$):

$$Pe = \frac{2R_0 v_0}{\kappa} \quad \text{and} \quad Nu_{circ}(Pe) = \frac{4}{\pi} Nu_p(Pe) \quad . \tag{18}$$

The local Nusselt number is $h_{circ}(\theta) = 2h_p(x)\sin\theta$, where $x = a(1+\cos\theta)/2$ (see Fig.2*b*).

Elliptical cylinder with semiaxes b_x *and* b_y. Similar procedure provides the Péclet and ower-all Nusselt numbers (based on $b_x + b_y$):

$$Pe = \frac{(b_x + b_y)v_0}{\kappa} \quad \text{and} \quad Nu_{ell}(Pe) = \frac{Nu_p(Pe)(b_x + b_y)}{P(b_x, b_y)}. \tag{19}$$

where $P(b_x, b_y)$ is the perimeter.

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