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Vacuum expectation values from fusion of vertex operators

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The algebra of fused vertex operators for the ABF model is defined and studied in the free fields approach. Vacuum expectation values of local operators in the scaling theory are reproduced from the matrix elements of the fused vertex operators.

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The algebraic approach to the two-dimensional exactly solvable models of the Statistical Mechanics [1, 2] turns out to be a very effective tool for finding integral representations for correlation functions and form-factors of local operators. In this approach a Hamiltonian of a model in the thermodynamic limit is diagonalized exactly by use of the so-called vertex operators (generators of a quadratic associative algebra). In the present work we extend the construction of the "Interaction Round a Face" type algebras [3 – 6] for the case of the fused vertex operators and apply it to a description of the local operators in a corresponding massive scaling theory. For definiteness, we elaborate the procedure for the Andrews, Baxter and Forrester (ABF) [7] integrable models of Statistical Mechanics. In what follows we identify the fused vertex operators with the operators inserting lattice spins. We argue that in the scaling limit [8, 9] the proposed bosonic operators determine vacuum expectation values of the local operators [10] of the theory. The last quantities, carrying all non-perturbative information on the theory, are of fundamental importance since they define both, short- and long- distance asymptotics [11] of the scaling correlation functions.

The fluctuation variables in the ABF models (and their non-unitary generalizations) are associated with cites of a two-dimensional square lattice and take integer values $1 \le k \le p'-1$. In the regime III there are $p-1$ ground states $1 \le l \le p-1$ in the model. The parameterization of the Boltzmann weights is given in terms of the elliptic theta functions $\theta_1(u|\tau)$ with the elliptic nome $e^{2\pi i\tau}$. We use the shorthand notation for the ratios of theta functions

$$[u] := \theta_1(\frac{\pi u}{\xi+1}|\frac{i\pi}{(\xi+1)\epsilon})/\theta_1(\frac{\pi}{\xi+1}|\frac{i\pi}{(\xi+1)\epsilon}).$$

Here $\xi := p/p'-p$, while $\epsilon > 0$ and $0 < u < 1$ are, respectively, the parameter measuring deviation from criticality and the spectral parameter. A local Boltzmann weight $\mathbf{U}$ is assigned to every configuration (m_a, m_b, m_c, m_d) of heights round a face with the sites (a, b, c, d). The non-vanishing Boltzmann weights satisfying the Yang-Baxter equation (YBE) are given (up to a gauge transform) as following

$$\mathbf{U}\begin{bmatrix} m\pm 2 & m\pm 1 \\ m\pm 1 & m \end{bmatrix} = R\,,$$
$$\mathbf{U}\begin{bmatrix} m & m\pm 1 \\ m\pm 1 & m \end{bmatrix} = R\,\frac{[m\pm u]}{[1-u][m]}\,, \qquad (1)$$
$$\mathbf{U}\begin{bmatrix} m & m\pm 1 \\ m\mp 1 & m \end{bmatrix} = R\,\frac{[\pm 1-m]}{[m]}\,\frac{[u]}{[1-u]}\,.$$

It is convenient to choose the factor R here to be such that the partition function per face in the thermodynamic limit equals to one

$$R(u) = e^{-\frac{\xi}{\xi+1}u\epsilon} \times$$
$$\times \exp\{\sum_{m=-\infty}^{\infty} \frac{\sinh m\epsilon \sinh \xi m\epsilon}{m \sinh(\xi+1)m\epsilon \sinh 2m\epsilon} e^{-2u\epsilon m}\}. \qquad (2)$$

The vertex operator algebra [2] is a quadratic graded associative algebra with the coefficients satisfying the YBE. The simplest case based on (1) has already been studied in Refs. [4, 5]. The solutions of the YBE which appear in the definition of new vertex operator algebras are essentially Boltzmann weights appearing in the construction of the fusion ABF models [12]. (Note that we use an obvious generalization for the case of rational values of the parameter ξ). For integers $0 < \nu \le \mu < p'-1$ let $a, b, c, d \in \{1, \ldots, p'-1\}$ satisfy the conditions $a-b,\ c-d\ = -\mu, -\mu+2, \ldots, \mu;\ a-d,\ b-c\ =$

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$-\nu, -\nu+2, \dots, \nu$. Starting from $\mathbf{U}$ in Eq. (1) we introduce the following notations

$$\mathbf{U}_{\nu\mu}\begin{bmatrix} a & b \\ d & c \end{bmatrix}\Big| u := \sum_{\{a_{i,j}\}} \prod_{i=0}^{\nu-1}\prod_{j=0}^{\mu-1} \mathbf{U}\begin{bmatrix} a_{i+1,j+1} & a_{i+1,j} \\ a_{i,j+1} & a_{i,j} \end{bmatrix}\Big|\tilde{u},$$

with $\tilde{u} := u + \frac{\mu-\nu}{2}\xi + (i-j)\xi$. The sum is taken over all values of $a_{i,j}$ allowed due to (1), while the variables $a_{0,j}, (0 \le j < \mu)$, $a_{i,\mu}, (0 \le i < \nu))$ are fixed. According to Ref. [12] the result depends only on the indexes $a = a_{\nu,\mu}$, $b = a_{\mu,0}$, $c = a_{0,0}$, $d = a_{0,\mu}$ and does not depends on $\{a_{0,j},\ j = 1, \dots, \mu-1\}$ and $\{a_{i,\mu},\ i = 1, \dots, \nu-1\}$. Though the expressions for the general solution of the YBE are rather complicated, our analysis depends basically on the properties of the $\mathbf{U}_{1\mu}$ fused weights which can be written out explicitly

$$\mathbf{U}_{1\mu}\begin{bmatrix} m'+1 & m+1 \\ m' & m \end{bmatrix}\Big| u = \frac{R_{1\mu}(u)}{[m][\mu-\bar{u}]}\times$$
$$\times [\frac{m+m'-\mu}{2}][\frac{m'-m+\mu}{2}-\bar{u}],$$
$$\mathbf{U}_{1\mu}\begin{bmatrix} m'-1 & m+1 \\ m' & m \end{bmatrix}\Big| u = \frac{R_{1\mu}(u)}{[m][\mu-\bar{u}]}\times$$
$$\times [\frac{m'-m+\mu}{2}][\frac{m'+m-\mu}{2}+\bar{u}],$$
$$\mathbf{U}_{1\mu}\begin{bmatrix} m'+1 & m-1 \\ m' & m \end{bmatrix}\Big| u = \frac{R_{1\mu}(u)}{[m][\mu-\bar{u}]}\times$$
$$\times [\frac{m-m'+\mu}{2}][\frac{m'+m+\mu}{2}-\bar{u}],$$
$$\mathbf{U}_{1\mu}\begin{bmatrix} m'-1 & m-1 \\ m' & m \end{bmatrix}\Big| u = \frac{R_{1\mu}(u)}{[m][\mu-\bar{u}]}\times$$
$$\times [\frac{m+m'+\mu}{2}][\frac{m-m'+\mu}{2}-\bar{u}].$$

Here $\bar{u} := u - \frac{\mu-1}{2}\xi$ and the factor $R_{1\mu}(u)$ is

$$R_{1\mu}(u) = R(u-\frac{\mu-1}{2}\xi)R(u-\frac{\mu-3}{2}\xi)\cdots R(u+\frac{\mu-1}{2}\xi).$$

We will use symmetric Boltzmann weights obtained from $\mathbf{U}_{\nu\mu}$ by applying the "gauge" transform

$$\mathbf{W}_{\nu\mu}\begin{bmatrix} a & b \\ d & c \end{bmatrix}\Big| u = \frac{\epsilon_b}{\epsilon_d}\left(\frac{(ab)_\mu(bc)_\nu}{(cd)_\mu(ad)_\mu}\right)^{\frac{1}{2}} \mathbf{U}_{\nu\mu}\begin{bmatrix} a & b \\ d & c \end{bmatrix}\Big| u.$$

Here $\epsilon_a = \pm 1$, $\epsilon_a\epsilon_{a+1} = (-1)^a$ and the symbols $(a,b)_\mu \equiv$ $\equiv (b,a)_\mu$ are defined via the theta functions

$$(a,b)_\mu = \left[\frac{\mu}{\frac{a-b+\mu}{2}}\right]^{-1} \frac{\left[\frac{a+b-\mu}{2}, (\frac{a+b+\mu}{2}\right]}{\sqrt{[a][b]}},$$

where $[a,b] := [a][a+1]...[b]$, $[a,a-1] = 1$. The set of functions $\mathbf{W}_{\nu\mu}$ satisfy the Yang-Baxter equation

$$\sum \mathbf{W}_{\nu\mu}\begin{bmatrix} a & s \\ e & f \end{bmatrix}\Big| u_1 \mathbf{W}_{\nu\tau}\begin{bmatrix} s & b \\ f & c \end{bmatrix}\Big| u_2 \mathbf{W}_{\tau\mu}\begin{bmatrix} e & f \\ c & d \end{bmatrix}\Big| u =$$
$$= \sum \mathbf{W}_{\nu\tau}\begin{bmatrix} a & f \\ f & e \end{bmatrix}\Big| u_2 \mathbf{W}_{\nu\mu}\begin{bmatrix} f & b \\ d & c \end{bmatrix}\Big| u_1 \mathbf{W}_{\tau\mu}\begin{bmatrix} a & s \\ f & b \end{bmatrix}\Big| u,$$

where $u := u_1 - u_2$ and the sum is taken over all allowed values of the variable f.

We check that the functions $R_{1\mu}(u)$ are subjects of the inversion relations $R_{1\mu}(-u)R_{1\mu}(u) = 1$ and

$$R_{1\mu}(u)R_{1\mu}(u+1) = - [u + \frac{\mu-1}{2}\xi]/[u+1-\frac{\mu-1}{2}\xi]. \qquad (3)$$

Using this property and Theorem 2.15 from Ref. [12] it is easy to verify that the weights $\mathbf{W}_{1\mu}$ satisfy the symmetry conditions

$$\mathbf{W}_{1\mu}\begin{bmatrix} a & b \\ d & c \end{bmatrix}\Big| u = \mathbf{W}_{1\mu}\begin{bmatrix} c & d \\ b & a \end{bmatrix}\Big| u, \qquad (4)$$
$$\mathbf{W}_{\mu 1}\begin{bmatrix} b & c \\ a & d \end{bmatrix}\Big| u = (-)^{\mu-1}\sqrt{\frac{[a][c]}{[b][d]}}\mathbf{W}_{1\mu}\begin{bmatrix} a & b \\ d & c \end{bmatrix}\Big| 1-u,$$

as well as the inversion relation

$$\sum_d \mathbf{W}_{\mu 1}\begin{bmatrix} a & d \\ b' & c \end{bmatrix}\Big| -u\, \mathbf{W}_{1\mu}\begin{bmatrix} a & b \\ d & c \end{bmatrix}\Big| u = \delta_{b,b'}. \qquad (5)$$

Heuristically the (so-called type I) operators in the Vertex Operator Approach [2] to the ABF model are defined as half infinite products of the Boltzmann weights $\mathbf{W}_{11}$ [2, 5]. It seems natural to generalize the construction for an extended operator algebra by implying the fusion procedure. We introduce the space $\mathcal{L}_{l,k}$ $(1 \le l \le p-1,\ 1 \le k \le p'-1)$ as a set of vectors $(a_0, a_1, a_2, \dots)$ such that $a_0 \equiv k$, $|a_j - a_{j+1}| = 1$ and the sequence $\{a_j\}$ stabilizes to $l, l+1, l, l+1, \dots$ for $j >> 1$. Let us define half infinite products of the fused Boltzmann weights $\mathbf{W}_{1,\mu}$

$$\Phi^{ab}_{1,\mu+1}(u)^{(a,a_1,a_2,\dots)}_{(b,b_1,b_2,\dots)} := \prod_{j=0}^{\infty} W_{1\mu}\begin{bmatrix} b_{j+1} & a_{j+1} \\ b_j & a_j, \end{bmatrix}\Big| u,$$

where $|b_j - b_{j+1}| = |a_j - a_{j+1}| = 1$, $a_0 = a$, $b_0 = b$, and treat it as an element of the half-infinite matrix $\Phi^{ab}_{1,\mu+1}(u)$ sending the vector $(b, b_1, b_2, \dots) \in \mathcal{L}_{l,b}$ to the vector $(a, a_1, a_2, \dots) \in \mathcal{L}_{l,a}$

$$\Phi^{(ab)}_{1,\mu+1} : \mathcal{L}_{l,b} \longrightarrow \mathcal{L}_{l,a}. \qquad (6)$$

The subtle point which we want to point out is that the matrix element of the "conjugate" operator

$$\Phi^{*ab}_{1,\mu+1}(u)^{(a,a_1,a_2,\dots)}_{(b,b_1,b_2,\dots)} := \prod_{j=0}^{\infty} \mathbf{W}_{1\mu}\left[\begin{matrix} a_j & b_j \\ a_{j+1} & b_{j+1}\end{matrix}\Big| u\right]$$

coincides with that one for $\Phi^{(ab)}_{1,\mu+1}$ due to Eq. (4). This requirement fixes a freedom in choosing a gauge transform.

The heuristic definition given above allows to derive the axioms for the vertex operator algebra. First, applying the YBE to a product of two operators we conclude that $\Phi_{1\mu+1}$ generate a quadratic associative algebra

$$\Phi^{(cb)}_{1,\mu+1}(u_2)\Phi^{(ba)}_{1,\nu+1}(u_1) = \\ = \sum_d \mathbf{W}_{\nu\mu}\left[\begin{matrix} a & d \\ b & c\end{matrix}\Big| u_{21}\right] \Phi^{(cd)}_{1,\nu+1}(u_1)\Phi^{(da)}_{1,\mu+1}(u_2). \quad (7)$$

Second, from the inversion relations (5) we find that vertex operators have to satisfy the normalization condition

$$\Phi^{(1,\mu+1)}_{1,\mu+1}(u)\Phi^{(\mu+1,1)}_{1,\mu+1}(u+1) = \frac{(-1)^{\mu-1}}{\sqrt{[\mu+1]}}. \quad (8)$$

In addition, it is assumed that the algebra can be extended by the so-called "Corner Hamiltonian" [1, 2, 5] element $H : \mathcal{L}_{lk} \longrightarrow \mathcal{L}_{lk}$ acting diagonally in the space of states. Its action on the vertex operators is defined as

$$e^{-2\epsilon v H}\Phi_{1\mu+1}(u)e^{2\epsilon v H} = \Phi_{1\mu+1}(u+v). \quad (9)$$

The last requirement which we impose is that the spectrum of D is described by the following formula [7]

$$Tr_{\mathcal{L}_{lk}}[q^H] = \chi_{lk}(q), \quad (10)$$

where $\chi_{lk}(q)$ is the character of the irreducible representation of the Virasoro algebra with the central charge and the highest weight vector being, respectively

$$c = 1 - \frac{6}{\xi(\xi+1)}, \quad \Delta_{lk} = \frac{((\xi+1)l - \xi k)^2 - 1}{4\xi(\xi+1)}. \quad (11)$$

To describe the matrix elements (and traces) of vertex operators satisfying properties (6)-(10) we use the free field technique originally developed in the CFT context [13 – 15] and generalized for the elliptic case in the works [3 – 6, 16]. Here we formulate main propositions on the properties of the bosonization procedure and add missing proofs.

Let us introduce the free bosonic field

$$\phi(u) = -\sqrt{\frac{\xi}{2(\xi+1)}}(\mathcal{Q} + 2i\epsilon u\mathcal{P}) + i\sum_{m\in Z/\{0\}} \frac{\beta_m}{m}e^{2\epsilon u m},$$

where the non-trivial commutation relations for the zero modes $\mathcal{Q}, \mathcal{P}$ and oscillators β_m are given as

$$[\mathcal{P}, \mathcal{Q}] = -i, [\beta_m, \beta_n] = m\frac{\sinh m\epsilon \sinh \xi m\epsilon}{\sinh 2m\epsilon \sinh(\xi+1)m\epsilon}\delta_{m+n}.$$

On the Fock module $\mathcal{F}_{lk} = \{\beta_{m_1}\cdots\beta_{m_N}|l,k>, m_1 \le \le \dots \le m_N < 0\}$ with the highest weight vector $|lk>$ the zero mode $\mathcal{P}$ acts as a number,

$$\omega|l,k> := \sqrt{2\xi(\xi+1)}\mathcal{P}|l,k> = \Big((\xi+1)l - \xi k\Big)|l,k>,$$

while $\beta_m|l,k> = 0, m > 0$. The grading operator in the Fock space is naturally defined as

$$D = \sum_{m=1}^{\infty} \frac{\sinh(\xi+1)m\epsilon \sinh 2m\epsilon}{\sinh m\epsilon \sinh \xi m\epsilon}\beta_{-m}\beta_m + \frac{\mathcal{P}^2}{2} - \frac{1}{24}.$$

The vertex operators will be expressed in terms of the exponents of the free bosonic field $V_\mu : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l,k+\mu}$ and $\bar{V} : \mathcal{F}_{l,k} \longrightarrow \mathcal{F}_{l,k-2}$

$$V_\mu(u) = \exp(i\sum_{j=0}^{\mu-1}\phi(u + \frac{\mu-2j-1}{2}\xi)), \\ \bar{V}(v) = \exp(-i\phi(v+\frac{1}{2}) - i\phi(v-\frac{1}{2})). \quad (12)$$

Define Lukyanov's screening operator as an integral operator

$$X_n(u) = \frac{[n]!}{n!}(\frac{-\epsilon}{\eta})^n \int \frac{dv_1}{\pi i}\cdots\frac{dv_n}{\pi i}\prod_{i<j}\frac{[v_i - v_j]}{[v_i - v_j - 1]} \times \\ \times \prod_{s=1}^{n}\frac{[v_s - u - \mu/2 - \omega - n + 1]}{[v_s - u - \mu/2]}\bar{V}(v_1)\cdots\bar{V}(v_n). \quad (13)$$

Here the integration is from u to $u + i\pi/\epsilon$ and η is a normalization constant which will be specified later. We claim that *acting in the direct sum of the Fock spaces* $\oplus\mathcal{F}_{lk}$ *the following screened vertex operators*

$$V^{(s)}_\mu(u) = X_s(u)V_\mu(u) : \mathcal{F}_{l,a} \longrightarrow \mathcal{F}_{l,a+\mu-2s} \quad (14)$$

satisfy the quadratic relations (7) but with the coefficients given by $\mathbf{U}_{\nu\mu}$.

Knowing the exact formulae for the bosonization (12)-(14) it is easy to derive matrix elements of the screened operators and analyze its properties. (It is convenient for us to work with integrals using the multiplicative variables $z = x^{2v}$, $x = e^{-\epsilon}$.) The oscillator modes contributions for the exponential operators

$$< V_\mu(0)V_\mu(u) >_{osc} = \rho^2_\mu g_\mu(e^{-2u\epsilon}),$$

$$< \bar V(0)V_\mu(u) >_{osc} = \rho_\mu\bar\rho\ w_\mu(e^{-2u\epsilon}),$$
$$< \bar V(0)\bar V(v) >_{osc} = \bar\rho^2\bar w(e^{-2u\epsilon}), \qquad (15)$$

are given as following

$$g_\mu(\zeta)= \frac{\{x^{2\xi+4}\zeta\}^2_\infty\{x^{2\xi+2+2\xi\mu}\zeta\}_\infty\{x^{2\xi+2-2\xi\mu}\zeta\}_\infty}{\{x^{2\xi+2}\zeta\}^2_\infty\{x^{2\xi+4+2\xi\mu}\zeta\}_\infty\{x^{2\xi+4-2\xi\mu}\zeta\}_\infty},$$
$$w_\mu(z)= \frac{(x^{1+(\mu+1)\xi}z)_\infty}{(x^{1-(\mu-1)\xi}z)_\infty},\ \bar w(z) = (1-z)\frac{(x^2z)_\infty}{(x^{2\xi}z)_\infty}.$$

Here and after we use the definitions for infinite products

$$(z)_\infty := \prod_{j=0}^{\infty}(1-zx^{2(\xi+1)j}),$$
$$\{z\}_\infty := \prod_{i,j,m=0}^{\infty}(1-zx^{2(\xi+2)i}x^{2\xi j}x^{4m}).$$

The constants ρ_μ and $\bar\rho$ enter the expression for matrix elements in the combinations ρ^2_μ and $\eta^{-1}\bar\rho$ only. The last ones turn out to be

$$\rho^2_\mu = g_\mu(1),\ \eta^{-1}\bar\rho = x^{\frac{\xi}{2(\xi+1)}}\frac{(x^{2(\xi+1)})_\infty}{(x^{2\xi})_\infty}.$$

Using these formulae, we find, for instance, that the two points matrix element of the vertex operators V_μ

$$G_\mu(e^{-2\epsilon u}) :=< 11|V^{(\mu)}_\mu(0)V^{(0)}_\mu(u)|11 >$$

defining the normalization of these operators is given in terms of the μ-fold integral

$$G_\mu(\zeta) = \left(\frac{\bar\rho x^{-\frac{\xi}{2(\xi+1)}}}{\eta(x^{2(\xi+1)})^2_\infty}\right)^\mu (\zeta x^{-2})^{\Delta_{1\mu}+1}g_\mu(\zeta)\rho^2_\mu \times$$
$$\times\frac{[\mu]!}{\mu!}\oint\cdots\oint\frac{dz_1}{2\pi i z_1}\cdots\frac{dz_\mu}{2\pi i z_\mu}\prod_{i\neq j}\frac{(z_i/z_j)_\infty}{(z_ix^{2\xi}/z_j)_\infty}\times$$
$$\times\prod_{s=1}^{\mu}\frac{E(z_sx^{-1+\xi(\mu+1)})}{(z_sx^{1-(\mu-1)\xi})_\infty(x^{1-(\mu-1)\xi}/z_s)_\infty}\times$$
$$\times\frac{E(z_sx^{1-\xi(\mu-1)}/\zeta)}{(z_sx^{1-(\mu-1)\xi}/\zeta)_\infty(\zeta x^{1-(\mu-1)\xi}/z_s)_\infty}, \qquad (16)$$

where $E(z) := (x^{2(\xi+1)})_\infty(z)_\infty(x^{2(\xi+1)}z^{-1})_\infty$. The integral can be transformed to the Askey-Roy type integral for which the explicit expression is known [17]. Applying the identity

$$g_\mu(\zeta)g_\mu(\zeta x^{-2}) = \prod_{s=0}^{\mu-1}\frac{(\zeta x^{-2\xi s})_\infty}{(\zeta x^{2\xi(s+1)})_\infty},$$

we arrive finally to the following elegant expression for the matrix element of two screened operators

$$G_\mu(x^{2u})= [\mu]!\ \epsilon_1\epsilon_{\mu+1}\frac{Q_\mu(x^{2u})}{Q_\mu(x^2)},$$

$$Q_\mu(\zeta)= \zeta^{\Delta_{1\mu}}g_\mu(\zeta)\prod_{s=0}^{\mu-1}\frac{(\zeta x^{2\xi(s+1)})_\infty}{(\zeta x^{2-2\xi s})_\infty}. \qquad (17)$$

It is obvious now that $G_\mu(x^2) = [\mu]!\ \epsilon_1\epsilon_{\mu+1}$.

Until now we concentrated on the operators acting in $\oplus\mathcal{F}_{l,k}(l,k\in Z)$. It is possible to prove that the bosonic fields (14), in fact, can be restricted to a smaller space, $\oplus\mathcal{L}_{l,k}$ $(1\le l\le p-1, 1\le k\le p'-1)$, the direct sum of irreducible representations of the deformed Virasoro algebra [4, 18, 5]. Let us recall its construction via a BRST reduction generalizing [15]. Consider the sequence of the maps

$$\cdots\xrightarrow{X^{(-1)}}\mathcal{F}_{l,k}\xrightarrow{X^{(0)}}\mathcal{F}_{l,-k}\xrightarrow{X^{(1)}}\mathcal{F}_{m,k-2p'}\xrightarrow{X^{(2)}}\cdots \qquad (18)$$

defined by the action of the appropriate screening charges $X_s(0)$, i.e.,

$$X^{(2j)}= X_k \quad : \mathcal{F}_{l,k-2jp'} \longrightarrow \mathcal{F}_{m,-k-2jp'},$$
$$X^{(2j+1)}= X_{p'-k} \ : \mathcal{F}_{l,-k-2jp'} \longrightarrow \mathcal{F}_{l,k-2(j+1)p'}.$$

It was proposed in Refs. [5, 6] that the following statements are true for $\epsilon > 0$

I. *The chain of maps (18) is a BRST complex, i.e.*

$$X^{(j)}X^{(j-1)} = 0. \qquad (19)$$

II. *The cohomologies of the complex turn out to be non-trivial only for* $j = 0$

$$Ker\ X^{(j)}\Big/\mathrm{Im}\ X^{(j-1)} = 0\ ,\ \text{if}\ j\neq 0. \qquad (20)$$

III. *Bosonic operators act in the spaces of cohomologies* $\mathcal{L}_{lk}\equiv Ker\ X^{(0)}\Big/\mathrm{Im}\ X^{(-1)}$. *(These spaces coincide with the irreducible representations of the Deformed Virasoro algebra [4, 18, 5]).*

Proof. The first statement has already been established in Ref. [16] basing on the fact that the screening operator X_n can be identified with X^n_1 acting on the correspondent space.

To prove the second statement, we follow words by words the line of the proof of Refs. [15] but for the deformed $\epsilon > 0$ case. Taking into account the results on the representation structure of the deformed Virasoro algebra [18], the only missing step would be that the screening charge $X^{(j)}$ is non-trivial. This, in turn, follows from the fact that for $1\le k\le p'-1$ and $l<0$ there exists a non-vanishing matrix element of the form

$$J =< l,-k|\beta^{s_1}_{n_1}\cdots\beta^{s_j}_{n_j}X^k|l,k >,$$

with $s_i\in N, 0<n_1<n_2<\cdots<n_j$ and $\sum s_in_i = -l$. From the commutation relations

$$[\beta_m,\bar V(v)] = -\frac{\sinh(\xi m\epsilon)}{\sinh(\xi+1)m\epsilon}z^m,$$

we show that the integrand of J contains the multiples $\sum_{i=1}^{j}(\sum_{t=1}^{k} z_t^{n_i})^{s_i}$ which form a basis in the space of symmetric polynomials of degree $|lk|$ in $z_1, \dots, z_k$. Therefore we can choose an element which would produce $(z_1 \cdots z_k)^{|l|}$. Using the expression for the couplings of exponential operators (15) we find that the problem is reduced to computation of the q-beta integral:

$$\oint \frac{dz_1}{2\pi i z_1} \cdots \frac{dz_k}{2\pi i z_k} \prod_{s=1}^{k} \frac{E(z_j x^{-1+2\xi(k-2s+1)})}{(z_j x^{-1})_\infty (x^{2\xi+3} z_j^{-1})_\infty} \times$$

$$\times \prod_{1\le a<b\le k} (1-\frac{z_a}{z_b}) \frac{(x^2 z_a/z_b)_\infty}{(x^{2\xi} z_a/z_b)_\infty} =$$

$$= \prod_{s=1}^{k} \frac{E(x^{2\xi s})}{E(x^{2\xi})} \frac{(x^{2(\xi+1)} x^{2\xi(s-1)})_\infty}{(x^{2\xi s})_\infty}, \quad (21)$$

which is non zero for $1 \le k \le p'-1$. Repeating the remaining arguments of [15] we verify the proposition.

To show that the vertex operators of the first type acts in the cohomologies of the Felder complex we have to show the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{F}_{l,k} & \xrightarrow{V_\mu^{(s)}} & \mathcal{F}_{l,k+\mu-2s} \\ X_k \downarrow & & \downarrow X_{k+\mu-2s} \\ \mathcal{F}_{l,-k} & \xrightarrow{V_\mu^{(\mu-s)}} & \mathcal{F}_{l,-k-\mu+2s} \end{array} \quad (22)$$

This can be proven from the $\mu = 1$ case using the commutation relations between X_k and V_1 derived in Ref. [19]

$$X_{k+1-2s}(u_0) V_1^{(s)}(u)\Big|_{\mathcal{F}_{l,k}} = V_1^{(1-s)}(u) X_k(u_0)\Big|_{\mathcal{F}_{l,k}}. \quad (23)$$

This completes the proof of the statements on the BRST complex.

We identify the lattice space of states $\mathcal{L}_{lk}$ with the cohomologies spaces of the Felder resolution. As for operators, we provide the restriction of the following bosonic operators

$$H\Big|_{\mathcal{L}_{l,a}} = D\Big|_{\mathcal{L}_{l,a}},$$

$$\Phi_{1,\mu+1}^{b,a}\Big|_{\mathcal{L}_{l,a}} = \frac{\epsilon_b}{\sqrt{(ab)_\mu}} V_\mu^{(a-b+\mu)/2}\Big|_{\mathcal{L}_{l,a}}. \quad (24)$$

Collecting the results described above, we finally are able to extend the statements of [5] to the following proposition for the $1 \le \mu \le p'-1$ case:

The bosonic operators in Eq. (14), (24) satisfy the commutation relations (7), (9), the normalization condition (8) and the character property (10).

In the rest of the paper, we would like to emphasize that the proposed bosonization construction for the vertex operators carries important information on the local operators of the scaling theory. Indeed, according to Ref. [20] the vacuum expectation values of the primary operators in the scaling $\epsilon \to 0$ limit of the ABF model should be defined as $< \hat\Phi_{1k} >= (-1)^{k-1} < 1k| \exp\left(-\pi^2 H/\epsilon\right) |1k>$, where $\exp(-\pi^2/2\epsilon)$ is a temperature-like parameter. We propose that the projection on the highest weight vectors $|1k>$ is a result of insertion of two vertex operators $\Phi_{1,k}$ in the matrix element $< 11| \cdots |11 >$. Let us note here that in the conformal case [14, 15] two copies of fused vertex operators of different chiralities are glued in a crossing invariant way to reconstruct operators with a trivial braiding, while the conjugation condition is satisfied automatically. In the off-critical case the local operator is constructed from the product of vertex operator and its conjugate. The commutativity of such pairs follows from the commutativity of fused transfer matrices, while the conjugation condition is provided by choosing the symmetric (4) expressions $W_{1\mu}$. In that way, the normalization of Boltzmann weights as well as the gauge transform are fixed in the local operators construction. Now, assuming the "conformal" normalization which is natural in a scaling theory

$$\lim_{\epsilon\to 0} < 11|\Phi_{1k}(0)\Phi_{1k}(u/\epsilon)|11 >= \frac{1}{(1-u)^{2\Delta_{1,k}}},$$

rather than the lattice one (8), and using Eq. (17), we find the following vacuum expectation values for scaling fields with the dimension Δ_{1k}

$$< \hat\Phi_{1k} >= (-1)^{k-1} (C\ M)^{2\Delta_{1k}} \lim_{\epsilon\to 0} Q_{k-1}(x^2)\sqrt{[k]}. \quad (25)$$

Here the k-independent proportionality coefficient C between the mass of kink M [9] and the temperature parameter $\exp(-\pi^2/2\epsilon)$ can be fixed from $k=3$ case due to Ref. [21]. Computing the limit (25) we arrive to the Lukyanov-Zamolodchikov answer for the vacuum expectation values of the scaling fields $\hat\Phi_{1k}$ in the conformal normalization [10, 22]

$$< \hat\Phi_{1k} >= (-1)^{k-1} \Big(M \frac{\sqrt{\pi}\Gamma(\frac{3}{2}+\frac{\xi}{2})}{2\Gamma(\frac{\xi}{2})}\Big)^{2\Delta_{1k}} \times$$

$$\times Q(1-(k-1)\xi). \quad (26)$$

The meromorphic function $Q(x)$ obtained from (17) can be rewritten as an analytic continuation of

$$Q(x) = \exp \int_0^\infty \frac{dt}{t} \left\{ \frac{\cosh 2t \sinh t(x-1) \sinh t(x+1)}{2 \cosh t \sinh t\xi \sinh t(1+\xi)} - \right. \left. - \frac{x^2-1}{2\xi(\xi+1)} e^{-4t} \right\}. \quad (27)$$

Finally, let us draw some concluding remarks. Our motivation for studying the fused vertex operators was that the integral representations for the correlation functions are rather difficult to be analyzed in the scaling limit. For a moment, we think that one of the most effective methods of studying correlation functions in the scaling theory [8, 9] is Al. Zamolodchikov's approach of combining results of the conformal perturbation and the form-factor theories. The most important objects in Ref. [11] are vacuum expectation values of local operators which determine normalizations of form factors as well as local OPEs. In the present paper we started studying a lattice origin of the vacuum expectation values by introducing and analyzing algebra of spin operators for the ABF model. Tracing the result (25)-(27) back to Eq. (17) we observe that the essential part of the Lukyanov-Zamolodchikov vacuum expectation values, the function $g_\mu(\zeta)$, appears in the fused vertex operators theory as a part of the normalization multiple of the Boltzmann weight $\mathbf{W}_{\mu\mu}$. The last one, in turn, is related with Baxter's partition function of the eight vertex model [1]. We checked that this phenomena is of general nature. It appears, for instance, for vacuum expectation values of local operators of other integrable models, like Z_N models [23] and perturbed W_n models [24, 25]. In that way, the formulated procedure proposes an alternative way for finding vacuum expectation values in scaling models for which the lattice vertex operators are known, e.g. [26, 27]. The nice feature of the construction is that the "reflection equations" [28, 22, 24] are automatically taken into account. It would be very interesting to generalize the construction to descendant operators [18], for which, in general, the reflection equations [28] are difficult to be solved, see e.g. [29, 30].

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