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LEVEL STATISTICS IN A METALLIC SAMPLE: CORRECTIONS TO THE WIGNER–DYSON DISTRIBUTION

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Deviation of the level correlation function in a mesoscopic metallic sample from the Wigner–Dyson distribution is calculated by using a combination of the renormalization group and non-perturbative treatment. For given spatial dimension the found correction is determined by the sample conductance.

The problem of level correlation in quantum systems attracts interest of physicists since the work of Wigner [1]. The random matrix theory developed by Wigner and Dyson [2] describes well the level statistics of various complex systems, like nuclei or atoms. Later on, Gor'kov and Eliashberg [3] put forward an assumption that the random matrix theory is also applicable to the problem of energy level correlations of a quantum particle in a random potential. This hypothesis was proven by Efetov, who showed [4] that the level-level correlation function $R(\omega)$ (its formal definition is given below) is described by the Wigner–Dyson distribution for $\omega \ll E_c$, E_c being the Thouless energy. For $\omega \gg E_c$, the behavior of the correlation function changes because the corresponding time scale ω^{-1} is smaller than the time E_c^{-1} the particle needs to diffuse through the sample. The form of the correlation function in this region is dependent on spatial dimensionality and was calculated in Ref.[5] by means of the diffusion perturbation theory.

In the present Letter we find a correction to the Wigner–Dyson distribution in the region $\omega \sim \Delta \ll E_c$, Δ being the mean level spacing. This is not a trivial task, because we calculate a correction to the result which is essentially non–perturbative.

We study the two–level correlation function $R(s)$ defined as

$$R(s) = \frac{1}{\langle \nu \rangle^2} \langle \nu(E)\nu(E+\omega)\rangle, \tag{1}$$

where $s = \omega/\Delta$, $\nu(E)$ is the density of states at the energy E and $\langle \ldots \rangle$ denote averaging over realizations of the random potential. As was shown by Efetov [4], the correlator (1) can be expressed in terms of a Green function of certain supermatrix σ-model. Depending on whether the time reversal and spin rotation symmetries are broken or not, one of three different σ-models is relevant, with unitary, orthogonal or symplectic symmetry group. We will consider the case of the unitary symmetry (corresponding to the broken time reversal invariance)

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throughout the paper; the results for two other cases will be presented at the end. The expression for $R(s)$ in terms of the σ-model then reads:

$$R(s) = \left(\frac{1}{4V}\right)^2 \mathrm{Re} \int DQ(r) \left[\int d^d r \mathrm{Str} Q\Lambda k\right]^2 \cdot$$
$$\cdot \exp\left\{-\frac{\pi\nu}{4} \int d^d r \mathrm{Str}[-D(\nabla Q)^2 - 2i\omega\Lambda Q]\right\}. \quad (2)$$

Here $Q = T^{-1}\Lambda T$ is 4×4 supermatrix, with T belonging to the coset space $U(1,1|2)$, $\Lambda = \mathrm{diag}\{1,1,-1,-1\}$, $k = \mathrm{diag}\{1,-1,1,-1\}$, Str denotes the supertrace, V is the system volume and D is the classical diffusion constant. To find the detailed description of the model and mathematical entities involved, a reader is refered to Refs.[4, 6].

For $\omega \ll E_c \sim D/L^2$ (L being the system size, so that $V = L^d$) the leading contribution to the integral (2) is given by the spatially uniform fields $Q(r) = Q$. Then the functional integral in eq.(2) reduces to an integral over a single supermatrix Q and can be calculated yielding the Wigner–Dyson distribution [4]:

$$R_{WD}(s) = 1 - \frac{\sin^2(\pi s)}{(\pi s)^2}. \quad (3)$$

The aim of the present paper is to calculate a correction to eq.(3) due to spatial fluctuations of $Q(r)$ in eq.(2). The procedure we are using for this purpose is as follows. We first decompose Q into the constant part Q_0 and the contribution $\tilde{Q}$ of higher modes with non–zero momenta. Then we use the renormalization group ideas and integrate out all fast modes. This can be done *perturbatively* provided the dimensionless (measured in units of $e^2/\hbar$) conductance $g = E_c/\Delta \gg 1$. As a result, we get an integral over the matrix Q_0 only, which has to be calculated *non-perturbatively*. We believe that this combination of the perturbative renormalization-group-type and non-perturbative treatment is of a methodological interest and might be used for other applications.

To begin with, we present the correlator $R(s)$ in the form

$$R(s) = \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial u^2} \int DQ \exp\{-\mathcal{F}(Q)\}|_{u=0} \ ;$$
$$\mathcal{F}(Q) = -\frac{1}{t}\int \mathrm{Str}(\nabla Q)^2 + \tilde{s}\int \mathrm{Str}\Lambda Q + \tilde{u}\int \mathrm{Str} Q\Lambda k \quad (4)$$

where $1/t = \pi\nu D/4$, $\tilde{s} = \pi s/2iV$, $\tilde{u} = \pi u/2iV$. Now we decompose Q in the following way

$$Q(r) = T_0^{-1}\tilde{Q}(r)T_0 \quad (5)$$

where T_0 is a spatially uniform matrix and $\tilde{Q}$ describes all modes with non-zero momenta. When $\omega \ll E_c$, the matrix $\tilde{Q}$ fluctuates only weakly near the origin Λ of the coset space. In the leading order, $\tilde{Q} = \Lambda$, thus reducing (4) to a zero-dimensional σ–model, which leads to the Wigner–Dyson distribution (3). To find the corrections, we should expand the matrix $\tilde{Q}$ around the origin Λ. This expansion starts as [4]

$$\tilde{Q} = \Lambda\left(1 + W + \frac{W^2}{2} + \ldots\right) \ , \quad (6)$$

where W is a supermatrix with the following block structure:

$$W = \begin{pmatrix} 0 & t_{12} \\ t_{21} & 0 \end{pmatrix} \quad . \tag{7}$$

Substituting this expansion into eq.(4), we get

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_0 + \mathcal{F}_1 + O(W^3) \ ; \\ \mathcal{F}_0 &= \int \mathrm{Str}\left[\frac{1}{t}(\nabla W)^2 + \tilde{s}Q_0\Lambda + \tilde{u}Q_0\Lambda k\right] \\ \mathcal{F}_1 &= \frac{1}{2}\int \mathrm{Str}[\tilde{s}U_0\Lambda W^2 + \tilde{u}U_{0k}\Lambda W^2] \end{aligned} \tag{8}$$

where $Q_0 = T_0^{-1}\Lambda T_0$, $U_0 = T_0\Lambda T_0^{-1}$, $U_{0k} = T_0\Lambda k T_0^{-1}$. Let us define $\mathcal{F}_{eff}(Q_0)$ as a result of elimination of the fast modes:

$$e^{-\mathcal{F}_{eff}(Q_0)} = e^{-\mathcal{F}_0(Q_0)}\langle e^{-\mathcal{F}_1(Q_0,W)}\rangle_W \tag{9}$$

where $\langle\ldots\rangle_W$ denote the integration over W. Expanding up to the order W^4, we get

$$\mathcal{F}_{eff} = \mathcal{F}_0 + \langle\mathcal{F}_1\rangle - \frac{1}{2}\langle\mathcal{F}_1^2\rangle + \frac{1}{2}\langle\mathcal{F}_1\rangle^2 + \ldots \tag{10}$$

The integral over the fast modes can be calculated now using the Wick theorem and the contraction rules [4, 7]:

$$\begin{aligned} &\langle \mathrm{Str}W(r)PW(r')R\rangle = \Pi(r-r')(\mathrm{Str}P\mathrm{Str}R - \mathrm{Str}P\Lambda\mathrm{Str}R\Lambda) \ ; \\ &\langle \mathrm{Str}[W(r)P]\mathrm{Str}[W(r')R]\rangle = \Pi(r-r')\mathrm{Str}(PR - P\Lambda R\Lambda); \\ &\Pi(r) = \int \frac{d^dq}{(2\pi)^d}\frac{\exp(iqr)}{\pi\nu Dq^2} \end{aligned} \tag{11}$$

where P and R are arbitrary supermatrices. The result is:

$$\begin{aligned} &\langle\mathcal{F}_1\rangle = 0 \ ; \\ &\langle\mathcal{F}_1^2\rangle = \frac{1}{2}\int dr dr' \Pi^2(r-r')(\tilde{s}\mathrm{Str}Q_0\Lambda + \tilde{u}\mathrm{Str}Q_0\Lambda k)^2. \end{aligned} \tag{12}$$

Substituting eq.(12) into eq.(10), we find

$$\begin{aligned} &\mathcal{F}_{eff}(Q_0) = \frac{\pi}{2i}s\mathrm{Str}Q_0\Lambda + \frac{\pi}{2i}u\mathrm{Str}Q_0\Lambda k + \frac{a_d}{16g^2}(s\mathrm{Str}Q_0\Lambda + u\mathrm{Str}Q_0\Lambda k)^2 \ ; \\ &a_d = \frac{1}{\pi^4}\sum_{\substack{n_1,\ldots,n_d=0 \\ n_1^2+\ldots+n_d^2>0}}^{\infty} \frac{1}{(n_1^2+\ldots+n_d^2)^2}. \end{aligned} \tag{13}$$

The value of the coefficient a_d depends on spatial dimensionality d. In particular, for $d = 1,2,3$ we have $a_1 = 1/90 \simeq 0.0111$, $a_2 \simeq 0.0266$, $a_3 \simeq 0.0527$ respectively. Using now eq.(4), we get the following expression for the correlator to the $1/g^2$ order:

$$R(s) \ = \ \mathrm{Re}\frac{1}{(2\pi i)^2}\int dQ_0\left\{\left(\frac{\pi}{2i}\right)^2(\mathrm{Str}Q_0\Lambda k)^2\left[1 - \frac{a_d}{16g^2}s^2(\mathrm{Str}Q_0\Lambda)^2\right] - \right.$$

$$- \frac{a_d}{8g^2}(\mathrm{Str}Q_0\Lambda k))^2 + \frac{\pi a_d}{8g^2 i}s(\mathrm{Str}Q_0\Lambda)(\mathrm{Str}Q_0\Lambda k)^2 \Big\} \cdot$$
$$\cdot \exp\{-\frac{\pi}{2i}s\mathrm{Str}Q_0\Lambda\}. \tag{14}$$

This integral over the supermatrix Q_0 can be calculated by using the known technique [4], yielding

$$R(s) = 1 - \frac{\sin^2(\pi s)}{(\pi s)^2} + \frac{a_d}{\pi^2 g^2}\sin^2(\pi s). \tag{15}$$

The last term in eq.(15) just represents the correction of order $1/g^2$ to the Wigner–Dyson distribution. The formula (15) is valid for $s \ll g$. Let us note that the smooth (non-oscillating) part of this correction in the region $1 \ll s \ll g$ can be found by using purely perturbative approach [5, 8]. For $s \gg 1$ the leading perturbative contribution to $R(s)$ is given by a two-diffuson diagram:

$$\begin{aligned} R_{pert}(s) - 1 &= \frac{\Delta^2}{2\pi^2}\mathrm{Re} \sum_{\substack{q_i = \pi n_i/L \\ n_i = 0,1,2,\dots}} \frac{1}{(Dq^2 - i\omega)^2} = \\ &= \frac{1}{2\pi^2}\mathrm{Re}\sum_{n_i \geq 0}\frac{1}{(-is + \pi^2 g n^2)^2}. \end{aligned} \tag{16}$$

At $s \ll g$ this expression is dominated by the $q = 0$ term, with other terms giving a correction of order $1/g^2$:

$$R_{pert}(s) = 1 - \frac{1}{2\pi^2 s^2} + \frac{a_d}{2\pi^2 g^2} \ , \tag{17}$$

where a_d was defined in eq.(13). This formula is obtained in the region $1 \ll s \ll g$ and is perturbative in both $1/s$ and $1/g$. It does not contain oscillations (which cannot be found perturbatively) and gives no information about actual small-s behavior of $R(s)$. The result (15) of the present Letter is much stronger: it represents the exact (non-perturbative in $1/s$) form of the correction in the whole region $s \ll g$.

The calculation presented above can be straightforwardly generalized to the other symmetry cases. The result can be presented in a form valid for all three cases:

$$R_\beta(s) = \left(1 + \frac{a_d}{2\beta\pi^2 g^2}\frac{d^2}{ds^2}s^2\right) R_\beta^{WD}(s), \tag{18}$$

where $\beta = 1(2,4)$ for the case of orthogonal (unitary, symplectic) symmetry; R_β^{WD} denotes the corresponding Wigner–Dyson distribution, explicit form of which can be found e.g. in [4, 9]. (We denote by g the conductance per one spin projection: $g = E_c/\Delta = \nu D L^{d-2}$, without multiplication by factor 2 due to the spin.)

For $s \to 0$ the Wigner–Dyson distribution has the following behavior:

$$\begin{aligned} &R_\beta^{WD} \simeq c_\beta s^\beta; \qquad s \to 0, \\ &c_1 = \frac{\pi^2}{6}; \ c_2 = \frac{\pi^2}{3}; \ c_4 = \frac{\pi^4}{135}. \end{aligned} \tag{19}$$

As is clear from eq.(18), the found correction does not change the power β, but renormalizes the prefactor c_β:

$$R_\beta(s) = \left(1 + \frac{(\beta+2)(\beta+1)}{2\beta}\frac{a_d}{\pi^2 g^2}\right) c_\beta s^\beta \quad ; \quad s \to 0. \tag{20}$$

The correction to c_β is positive, that means physically a weakening of the level repulsion.

In conclusion, we have calculated the deviation of the level-level correlation function $R_\beta(s)$ in a mesoscopic metallic sample from its universal Wigner–Dyson form, using the supersymmetric sigma-model approach. The found correction is of order $1/g^2$, where g is the dimensionless conductance. It does not change the power β of the s^β behavior of the correlator $R(s)$ as $s \to 0$, but renormalizes the corresponding prefactor.

To get this result, we developed a novel method of calculation which combines perturbative elimination of fast diffusive modes (in spirit of renormalization group ideas) and subsequent non-perturbative evaluation of an integral over the zero mode. In the present paper we have used it to find the eigenvalues correlation function, but it may have other applications. In particular, deviation of eigenfunctions statistics in diffusive regime from the random matrix theory predictions can be successfully studied in this way [10].

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