Pis'ma v ZhETF, vol. 77, iss. 12, pp. 769 – 771 © 2003 June 25

Vacuum energy and Universe in special relativity

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Submitted 14 May 2003

The problem of cosmological constant and vacuum energy is usually thought of as the subject of general relativity. However, the vacuum energy is important for the Universe even in the absence of gravity, i.e. in the case when the Newton constant G is exactly zero, $G = 0$. We discuss the response of the vacuum energy to the perturbations of the quantum vacuum in special relativity, and find that as in general relativity the vacuum energy density is on the order of the energy density of matter. In general relativity, the dependence of the vacuum energy on the equation of state of matter does not contain G, and thus is valid in the limit $G \to 0$. However, the result obtained for the vacuum energy in the world without gravity, i.e. when $G = 0$ exactly, is different.

PACS: 03.30.+p, 04.90.+e, 98.80.−k

The problem of the vacuum energy appears to be more general than the cosmological constant problem which arises in general relativity [1]. Earlier we discussed the vacuum energy and its relation to the cosmological constant considering general relativity as an effective theory [2, 3]. We found that the energy of the equilibrium vacuum state at zero temperature is zero due to the general thermodynamic Gibbs-Duhem relation applied to the vacuum as a medium. The non-zero value of the vacuum energy, or more exactly the gravitating part of the vacuum energy, comes from perturbations of the vacuum state. In typical situations the perturbations of the vacuum are caused by the gravitating matter, and thus the induced vacuum energy density must be on order of the energy density of matter, which results in the cosmological constant consistent with observations [4].

Now we extend the discussion of the vacuum energy to the case of special relativity, i.e. to the Universe without gravity. Though in the world without gravity the cosmological constant is absent, the vacuum energy still plays an important role in the structure of the Universe. We find how the vacuum energy responds to matter in special relativity, and how this allows us to stabilize the static special-relativity Universe filled with matter having arbitary equation of state.

The cosmological term in the action for the general relativity is

$$S_\Lambda = -\int d^4x\sqrt{-g}\rho^{\rm vac}. \tag{1}$$

The corresponding stress-energy tensor of the vacuum is obtained by variation of the action over the metric tensor $g^{\mu\nu}$

$$T^{\rm vac}_{\mu\nu} = \rho^{\rm vac}u_\mu u_\nu + P^{\rm vac}(u_\mu u_\nu - g_{\mu\nu}) = \rho^{\rm vac}g_{\mu\nu}. \tag{2}$$

Here $\rho^{\rm vac}$ is the vacuum energy density and u^μ the 4-velocity of the vacuum. Since the equation of state for the vacuum is

$$P^{\rm vac} = -\rho^{\rm vac}, \tag{3}$$

the energy-momentum tensor does not depend on the 4-velocity u^μ, and thus is the same in any coordinate system.

The Eq.(2) with equation of state (3) are valid for the vacuum in special relativity too, i.e. in the absence of the dynamical field $g^{\mu\nu}$. These equations are obtained by the conventional procedure used in quantum field theory, when one introduces the fictituous field, such as the fictituous gauge fields or fictituous metric, and calculate the response of the vacuum to these fields. Moreover, the equation of state $P^{\rm vac} = -\rho^{\rm vac}$ is even more general, since it is valid even in the non-relativistic theories, where the quantum vacuum is played by the ground state of the quantum condensed matter. This equation of state comes from the general thermodynamic Gibbs-Duhem relation applied to the homogeneous ground state of a condensed matter (see. e.g. [3]).

Let us consider the Universe in special relativity (i.e. in the absence of gravity, $G = 0$), which is filled with non-gravitating homogeneous matter – the perfect cosmic fluid, and discuss how the vacuum responds to the matter and stabilizes this Universe. The energy-momentum tensor of matter is

$$T^{\rm M}_{\mu\nu} = \rho^{\rm M}v_\mu v_\nu + P^{\rm M}(v_\mu v_\nu - g_{\mu\nu}), \tag{4}$$

where v^μ the 4-velocity of matter, and $\rho^{\rm M}$ and $P^{\rm M}$ are energy density and pressure of matter in the comoving frame.

In the general coordinate frame, the energy and momentum density of matter are

$$\tilde{\rho}^{\rm M} = \frac{\rho^{\rm M} + \frac{v^2}{c^2} P^{\rm M}}{1 - \frac{v^2}{c^2}}, \quad \mathbf{p}^{\rm M} = \frac{\mathbf{v}}{c^2} \frac{\rho^{\rm M} + P^{\rm M}}{1 - \frac{v^2}{c^2}}. \quad (5)$$

The obvious consequence of Eq.(5) is that the energy and the momentum of matter do not satisfy the relativistic relation between the energy and momentum

$$\mathbf{P} = E \frac{\mathbf{v}}{c^2}. \quad (6)$$

The reason for that is related to the external forces acting on matter, which violate the Lorentz invariance, since they establish the preferred reference frame in which these forces are isotropic. These forces are presented in Eq.(5) through the pressure $P^{\rm M}$ of matter, which is supported by the external pressure (see Sec. 14 of Ref. [5]).

If the Universe is completely isolated from the "environment", the external pressure is absent, $P^{\rm M} = P_{\rm external} = 0$, and the Lorentz-invariant equation (6) is restored. But the typical matter considered in cosmology, such as the relativistic plasma, does not exist at zero pressure as an equilibrium state, except for the extreme limit case of the cold matter. Thus, within the special relativity the Universe must be either empty, or contain such a matter which can exist in equilibrium at zero pressure (the matter in a cold liquid state, for example).

The vacuum gives an alternative scenario for the equilibrium static Universe with matter to exist in special relativity. The equilibrium state is achieved when the pressure of the cosmological matter is compensated by the partial pressure of the vacuum, so that the external pressure becomes zero:

$$P_{\rm external} = P^{\rm vac} + P^{\rm M} = 0, \quad (7)$$

and the Universe (matter + vacuum) can be in equilibrium without external environment. For this equilibrium Universe the Eq.(6) for the energy and momentum of the whole Universe is also restored.

Using equation of state (3) and the equilibrium condition (7) one obtains the density of the vacuum energy induced by matter with pressure $P^{\rm M}$ in the equilibrium Universe:

$$\rho^{\rm vac} = P^{\rm M}. \quad (8)$$

Since the vacuum momentum is zero

$$\mathbf{p}^{\rm vac} = 0, \quad (9)$$

the total energy density and momentum density of the system (matter + vacuum) become

$$\rho_{\rm total} = \tilde{\rho}^{\rm M} + \rho^{\rm vac} = \frac{\rho^{\rm M} + \rho^{\rm vac}}{1 - \frac{v^2}{c^2}},$$
$$\mathbf{p}_{\rm total} = \mathbf{p}^{\rm M} + \mathbf{p}^{\rm vac} = \frac{\mathbf{v}}{c^2} \frac{\rho^{\rm M} + \rho^{\rm vac}}{1 - \frac{v^2}{c^2}}. \quad (10)$$

They satisfy the relativistic equation (6), and the corresponding density of the rest mass of the system is

$$\rho_{\rm total} = \frac{m_{\rm rest} c^2}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad m_{\rm rest} = \frac{\rho^{\rm M} + \rho^{\rm vac}}{c^2 \sqrt{1 - \frac{v^2}{c^2}}}. \quad (11)$$

The extra factor $\sqrt{1 - v^2/c^2}$ in the denominator of the rest mass is cancelled by the relativistic transformation of the volume: the volume V in the frame of the measurement and the volume $V_{\rm comoving}$ in the comoving frame are related as $dV = dV_{\rm comoving} \sqrt{1 - v^2/c^2}$, so that the total rest energy of the system is

$$M_{\rm rest} = \int dV \frac{\rho^{\rm M} + \rho^{\rm vac}}{\sqrt{1 - \frac{v^2}{c^2}}} = V_0 \left(\rho^{\rm M} + \rho^{\rm vac}\right). \quad (12)$$

Thus the whole world represents the relativistic object whose rest mass is the sum of the rest energies of the matter and quantum vacuum. The energy density of the quantum vacuum induced by the nongravitating ($G = 0$) matter is completely determined by the equilibrium condition (7) and equation of state for matter $P^{\rm M} = w^{\rm M} \rho^{\rm M}$:

$$\rho^{\rm vac}_{G=0} = w^{\rm M} \rho^{\rm M}. \quad (13)$$

We can compare the vacuum energy (13) in the Universe in special relativity, i.e. when $G = 0$, with the vacuum energy in Universes in general relativity, i.e. when $G \neq 0$. For the Einstein static closed Universe [6, 3] the vacuum energy induced by the gravitating matter is

$$\rho^{\rm vac}_{\rm Einstein} = \frac{1}{2} \left(1 + 3w^{\rm M}\right) \rho^{\rm M}, \quad (14)$$

and for the Gödel steady-state rotating Universe [7, 3] it is:

$$\rho^{\rm vac}_{\rm Goedel} = -\frac{1}{2} \left(1 - w^{\rm M}\right) \rho^{\rm M}. \quad (15)$$

In all three Universes, the density of the vacuum energy induced by matter is proportional to the energy density of matter. Equations (14) and (15) for the worlds with gravity do not depend on the Newton constant G, and thus are valid in the limit $G \to 0$. But they do not coincide with Eq.(13) for the world without gravity, i.e. when G is exactly zero. While in special relativity the vacuum response to matter is determined by the condition of zero external pressure, $P_{\rm external} = 0$, in the case of the gravitating matter the condition of gravineutrality is added [3]. The pressure and energy of the gravitational field contributes to both conditions even in the limit $G \to 0$. These contributions come from the space curvature in the Einstein Universe and from the local rotational metric in the Gödel Universe.

As distinct from the static gravitating Universes which experience different types of the gravitational instabilities, in special relativity the Universe with matter is stable, if the quantum vacuum itself is the stable object. In the latter case, small perturbations of the vacuum state, caused by cold or hot matter, do not destabilize the system. This fact is well known for the condensed matter analogs of the Universe – quantum liquids, where the role of the quantum vacuum is played by the superfluid condensate, and the role of the relativistic matter is played by the "relativistic" quasiparticles, with c being the maximum attainable speed of the low-energy quasiparticles. Examples are provided by the Bose-superfluid ^{4}He and Fermi-superfluid ^{3}He-A. Both quantum liquids are stable at $T = 0$ and $P = 0$, and their stability is not violated by the massless "relativistic" quasiparticles which appear at $T \neq 0$ forming the analog of matter in these toy Universes. For both liquids the equation (13) is valid with the "relativistic" equation of state $w^{\rm M} = 1/3$, if the liquids are isolated from the environment [2]. Though in both quantum liquids there are the low-frequency collective modes corresponding to the dynamics of the effective metric, this effective gravity does not obey the general covariance and is not Newtonean at large distances. As a result the effective gravity in these liquids does not modify the special-relativistic equation (13).

For such condensed matter systems, the relativistic equations (10) obtained for the energy and momentum of a Universe in special relativity are also applicable, but with one reservation. As distinct from its special relativity counterpart, the quantum vacuum (condensate) in condensed matter does not obey the effective Lorentz invariance obeyed by the excitations of the condensate – quasiparticles. In particular, the momentum density of the quantum condensate is non-zero. As distinct from Eq.(9), the superfluid condensate moving with the so-called superfluid velocity $\mathbf{u}_s$ carries the momentum density $\mathbf{p}^{\rm vac} = mn\mathbf{u}_s$, where m is the mass of particles comprizing the condensate (atoms of liquid), and n is their number density. As a result, for quantum liquids the relativistic equations are valid only in the reference frame moving with the condensate. The full correspondence between the quantum vacuum and superfluid condensate could occur only for such hypothetical condensates whose "atoms" are massless, $m = 0$. However, the difference between the relativistic quantum vacuum and non-relativistic quantum condensate does not change the conclusion that the vacuum response stabilizes the non-gravitating Universe.

I thank A.A. Starobinsky for fruitful discussions. This work was supported by ESF COSLAB Programme and by the Russian Foundations for Fundamental Research.

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