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A stable static Universe?

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Starting from the assumption that general relativity might be an emergent phenomenon showing up at low-energies from an underlying microscopic structure, we re-analyze the stability of a static closed Universe filled with radiation. In this scenario, it is sensible to consider the effective general-relativistic configuration as in a thermal contact with an "environment" (the role of environment can be played, for example, by the higher-dimensional bulk or by the trans-Planckian degrees of freedom). We calculate the free energy at a fixed temperature of this radiation-filled static configuration. Then, by looking at the free energy we show that the static Einstein configuration is stable under the stated condition.

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1. Introduction. The development of a geometrical description of the gravitational field led Einstein in 1917 to propose that the Universe could be, in the overall, a three-dimensional sphere with no other evolution than that provided by local physics [1]. To make this sort of equilibrium state for the Universe compatible with his geometrical field equations, he introduced the afterward famous cosmological constant. The cosmological constant succeeds in counterbalancing the collapsing tendency of all the matter in the Universe. Later, in 1930, Eddington proved that Einstein's static Universe was unstable under homogeneous departures from the equilibrium state [2]. At that time, the recession of galaxies had already been observed by Hubble. For this reason and in view of the instability, the Einstein model was considered as a possible initial state for the Universe that once destabilized would start to expand. (This point of view has been revisited recently in [3, 4]).

Eddington did not clearly analyze what could trigger the development of the instability, but vaguely associated it with the formation of condensations. Thanks to a very interesting series of works [3 – 8] (we do not intend to be exhaustive), at present we know that the instability of models of the Universe that are closed, and homogeneous, isotropic and static on the overall, is a much more subtle issue. Here on we will call all these models Einstein models, independently of their matter content, although the model proposed by Einstein himself corresponds to a Universe filled with dust.

On the one hand, Harrison showed [7] that if the equation of state of matter is such that its associated speed of sound c_s is greater than $1/\sqrt{5}$ all the physical inhomogeneous perturbations are neutrally stable. This has been recently emphasized and extended by Ellis and Maartens [3] and Barrow et al. [4]. This case includes a radiation dominated universe. For $0 \neq c_s < 1/\sqrt{5}$ only a finite set of inhomogeneous modes become unstable. What happen is that in a finite-size Universe, as an Einstein model, the Jeans scale for the formation of condensations is a significant fraction of the maximum attainable scale. Therefore for higher enough speeds of sound, only the Universe as a whole could develop an instability. For this same condition, $c_s > 1/5$, Gibbons also showed that the Einstein point corresponds to a local maximum of the entropy among the set of geometries conformally related to it that have a moment of time symmetry [8].

On the other hand, Bonnor showed [6] that, at least in the simplest case in which matter satisfies an equation of state, in order to really depart from the Einstein state one would need a global decrease in pressure in the entire Universe, (this process had already been discussed by Lemaȋtre in 1931 under the name of "stagnation" [5]). This suggests that a static Universe describable in the cosmological scales as filled with dust, for which $p = 0$, could not be able to change its static global state, but only develop instabilities on smaller scales. On the other extreme, a static Universe filled with radiation could in principle exit from this state towards a Friedman expansion by decreasing its pressure. Here we will concentrate on this later model and its instability.

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In this letter, we will analyze the stability of a closed and static Universe filled with radiation, but starting from notions somewhat different from those in standard general relativity. General relativity is commonly considered to be a low-energy effective theory that emerges from a deeper underlying structure. A particular realization of this situation is suggested by the gravitational features showing up in many condensed matter systems (such as liquid Helium) in the low-energy corner [9]. Theses type of systems suggest that both, matter particles and interaction fields, could be different emergent features of the underlying system: they will correspond to quasiparticles and collective-field excitations of a multi-particle quantum system. For example, in the phase A of ^{3}He, the quasiparticles correspond to Weyl fermions and the collective fields to electromagnetic and gravitational (geometrical) fields.

Imagine now that a Universe of the Einstein type was the effective result of describing the geometric and matter-like degrees of freedom emerging from the underlying structure. A photons-filled Einstein Universe will have a specific temperature. In the standard general relativity, the stability of the system is analyzed under the assumption of adiabaticity: there is no heat transfer in or out the Universe because "there is nothing outside the Universe". However, in the emergent picture described above there is not any a priory reason to consider the system as effectively thermodynamically closed (let us remark that this is a non standard general relativitic behaviour). Therefore, it is natural to ask what would happen when perturbing the Einstein state if the temperature of the underlying structure stays constant. We will not enter on what sets and controls this temperature, but only assume that it is independent from the behaviour of the effective Universe. Let us emphasize that we do not disturb the general relativity, so that the solution for the equilibrium static Universe is the same that follows from the Einstein equations. We only allow for the heat exchange with the "environment".

Apart from the emergent gravity picture, there exist other situations in which the image of an externally fixed temperature might also have sense. These are situations in which the 4-dimensional world of standard general relativity does not conform a completely closed system. We can imagine, for example, scenarios with extra dimensions, (playing the role of environment), from which energy can flow in and out the 4-dimensional section.

In the following we will compare Eddington's stability analysis with an analysis based on the fact that the temperature of the system is kept fixed. For that, we will calculate the free energy of radiation dominated Einstein states. Let us start now by reviewing the standard Eddington instability argument.

2. Eddington's instability analysis. Consider a generic positive-curvature FRW metric written in the form

$$ds^2 = -N^2(t)dt^2 + a^2(t)\Omega_{ij}dx^i dx^j. \tag{1}$$

Here $N(t)$ is the lapse function, $a(t)$ the scale factor and Ω_{ij} the metric on a unit three sphere. Following Schutz [10], the Einstein equations for a Universe filled with a perfect fluid, can be obtained by varying the action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R-2\Lambda) + \\ + \int d^3x\sqrt{-g}\,K + \int d^4x\sqrt{-g}\,p. \tag{2}$$

This is the standard Einstein-Hilbert action supplemented with a boundary term and the volume integral of the fluid pressure p. To obtain the standard form of the Friedman and Raychaudhuri equations we can substitute the previous FRW ansatz in the action and, after variation, set the lapse function equal to unity, $N=1$. Specifically the action can be written as

$$S = \frac{2\pi^2}{8\pi G}\int dt\; Na^3\;\left[3\left(-\frac{\dot a^2}{a^2}\frac{1}{N^2}+\frac{1}{a^2}\right)-\Lambda\right] + \\ + 2\pi^2\int dt\; Na^3\; p(N). \tag{3}$$

Here, the explicit dependence of the pressure on its argument is determined by the condition

$$N\frac{\partial p(N)}{\partial N} = -(\rho+p). \tag{4}$$

For example, for a radiation equation of state $\rho = 3p$ one obtains from this condition that $p = CN^{-4}$ with C a constant.

By looking at the previous action and having in mind that we are interested on the analysis of the static solutions of the system, we can define a different and simpler functional containing all the relevant information:

$$S_{\rm st} = \int dt\; Na^3\;\left[\frac{3}{a^2} - \Lambda + \tilde p(N)\right]. \tag{5}$$

Here, we have rescaled the density and pressure as $\tilde\rho = 8\pi G\rho$, $\tilde p = 8\pi G p$. We can easily see that by varying with respect to N and a and setting $N=1$ we obtain

$$(\Lambda + \tilde\rho)a^2 = 3, \tag{6}$$

$$(\Lambda - \tilde p)a^2 = 1. \tag{7}$$

In the case of a Universe filled with radiation, $\tilde\rho = 3\tilde p$, these relations give us the radiation Einstein conditions $\tilde\rho = \Lambda$, $a_0^2 = (3/2)\Lambda^{-1}$.

By looking at the functional $S_{\rm st}$ (setting $N=1$), one can also see that the Einstein point is not stable. Taking into account that the kinetic term for the scale factor enter the gravitational action (3) with a negative sign, the local maxima of the functional $S_{\rm st}$ will correspond to unstable points. This is just the case for the Einstein point (one can perform explicitly the second variation with respect to a to check this local behaviour). This is the Eddington instability we described in the introduction. We have obtained it here in this variational way because of later convenience in comparing it with the result obtained from the free energy function.

3. Free energy of a static gravitational configuration. Let us now take a completely different point of view. Let us analyze what happen when the perturbation to the Einstein model is performed as immersed in a thermal reservoir at a fixed temperature. For that let us consider the free energy of static gravitational configurations. We follow the treatment described in [11] and references there in. The free energy of the purely gravitational part of a static configuration is determined by the Euclidean action of the configuration (see for example [12]):

$$F_0 = T_0 I = -\frac{T_0}{16\pi G}\int d\tau d^3x\sqrt{g_e}\,(R_e - 2\Lambda). \tag{8}$$

Here the temporal coordinate τ is periodic with period equal to the inverse of the temperature T_0. The symbols R_e and g_e stand respectively for the Euclidean curvature and Euclidean metric of the configuration. We can realize that this term coming from the purely gravitational sector do not actually depend on the temperature so they will be there even at zero temperature

$$F_0 = -\frac{1}{16\pi G}\int d^3x\sqrt{g_e}\,(R_e - 2\Lambda). \tag{9}$$

We are assuming that a proper Einstein-Hilbert behaviour is emerging in the low-energy corner. Let us remind you that this is not what normally happen in the standard condensed matter systems we know of. In these cases the Einstein-Hilbert behaviour is supplemented with non-covariant terms (see [9]).

Let us now consider the free energy of a gas of photons (radiation) inside a curved but static geometry. The leading term in the temperature on the free energy function is

$$F_1 = -\frac{\sigma}{3}\int d^3x\sqrt{g_e}T^4(x), \tag{10}$$

where $\sigma \equiv \pi^2 k_B^4/15\hbar^3c^2$ is the Stefan-Boltzmann constant, and

$$T(x) = \frac{T_0}{\sqrt{g_{00}(x)}} \tag{11}$$

the Tolman temperature; (we will see later that there are other contributions to the free energy in lower powers of the temperature). For the particular geometries we are interested in here, the total free energy can be written as

$$F(a,N,T) = F_0 + F_1 = \\ = \frac{2\pi^2 a^3}{8\pi G}\left[-\frac{3N}{a^2} + N\Lambda - \frac{\tilde p}{N^3}\right], \tag{12}$$

with

$$\tilde p = \frac{1}{3}\tilde\rho = \frac{8\pi G}{3}\sigma T_0^4 : \text{const.} \tag{13}$$

From this free energy, associated with a static geometry filled with radiation at a temperature T_0, we can obtain the Einstein static condition. It corresponds to the one that extremises the function F. Variation with respect to N with an afterwards evaluation in $N=1$ gives

$$(\Lambda + 3\tilde p)a^2 = (\Lambda + \tilde\rho)a^2 = 3. \tag{14}$$

Variation with respect to a yields

$$(\Lambda - \tilde p)a^2 = 1. \tag{15}$$

Therefore we have found the same expressions that before: the conditions for a static Einstein Universe filled with radiation.

By inspection of the free energy function we can see that now the Einstein static point is located at a local minimum. This is the main point we want to highlight in this letter. *If the perturbation of the radiation filled Einstein Universe were done under the influence of an externally fixed temperature, (something outside the realm of standard general relativity) then, the Einstein point will be stable.*

4. Corrections due to the temperature. In the previous section we analyzed the free energy of a system composed by static geometries of the Einstein type, as the containers, plus photon gases, as the contents. The free energy (12) is an approximation as it does not contain additional contributions in smaller powers of the temperature. In the high-temperature limit $T^2 \gg \hbar^2 R_e$, the total free energy for a gas of photons in a static spacetime is [11, 13]

$$F = -\frac{1}{16\pi G}\int d^3x\sqrt{g_e}\,(R_e - 2\Lambda) - \\ - \frac{\sigma}{3}\int d^3x\sqrt{g_e}T^4 + \bar\sigma\int d^3x\sqrt{g_e}T^2[R_e + 6\omega^2] \tag{16}$$

with

$$\omega_\mu = \frac{1}{2}\partial_\mu \ln |g_{00}(r)|. \quad (17)$$

Here the prefactor $\bar{\sigma}$ in the $T^2 R_e$ term is $\bar{\sigma} = N_v/36\hbar$ and is obtained by integration over thermal photon fields (see [14]). If the integration had been made using minimally coupled scalar fields the coefficient would have been $\bar{\sigma} = -N_s/144\hbar$ with N_s the number of minimally coupled scalar fields; equivalently for N_d Dirac fermions one would have $\bar{\sigma} = N_d/144\hbar$ [11, 13]. In our particular case, this free energy yields

$$F(N, a, T_0) = = \frac{2\pi^2 a^3}{8\pi G}\left[-\frac{3N}{a^2} + N\Lambda - \frac{\tilde{p}}{N^3} + 8\pi G\bar{\sigma}\frac{6T_0^2}{Na^2}\right], \quad (18)$$

or expressing everything in terms of $\tilde{\rho}$ and denoting the constant factor $(8\pi G/\sigma)^{1/2} 6\bar{\sigma}$ by the letter b,

$$F(N, a, T_0) = = \frac{2\pi^2 a^3}{8\pi G}\left[-\frac{3N}{a^2} + N\Lambda - \frac{\tilde{\rho}}{3N^3} + b\,\frac{\tilde{\rho}^{1/2}}{Na^2}\right]. \quad (19)$$

Varying with respect to N and a we find now

$$(\Lambda + \tilde{\rho})a^2 - b\tilde{\rho}^{1/2} = 3, \quad (20)$$

$$\left(\Lambda - \frac{1}{3}\tilde{\rho}\right) a^2 + \frac{1}{3} b\tilde{\rho}^{1/2} = 1. \quad (21)$$

Manipulating these two conditions one obtains

$$\Lambda = \frac{3}{2a^2} = \frac{\tilde{\rho}}{1 + \frac{2}{3} b\tilde{\rho}^{1/2}}. \quad (22)$$

Note that in emergent gravity the external temperature and thus $\tilde{\rho}$ is fixed, while the cosmological constant Λ (i.e. the vacuum pressure) is adjusted to the thermodynamic equilibrium. This is the reason why in the emergent gravity the cosmological constant is always much smaller than its 'natural' Planck value: in our case $\Lambda \sim T_0^4/E_{\rm Planck}^2 \ll E_{\rm Planck}^2$.

In the particular case of photon gas with $\bar{\sigma} = 1/36\hbar$ one obtains the following modification of the Einstein point by thermal fluctuations (here we use $\hbar = c = 1$):

$$\Lambda = \frac{3}{2a^2} = \frac{8}{15}\frac{\pi^3 G T_0^4}{1 + (8/9)\pi G T_0^2}\,. \quad (23)$$

This corresponds to the original Einstein point with the Newton constant renormalized by thermal fluctuations:

$$\Lambda = \frac{3}{2a^2} = \frac{8}{15}\pi^3 \tilde{G} T_0^4, \quad \tilde{G}^{-1} = G^{-1} + \frac{8}{9}\pi T_0^2. \quad (24)$$

For $GT_0^2 \ll 1$, i.e. when $T_0^2 \ll E_{\rm Planck}^2$, the result in Eq.(23) coincides with that obtained by Altaie and Dowker, see Eqs.(44) and (41) in [14]:

$$\frac{1}{a^2} = \frac{16}{45}\pi^3 G T_0^4 \left(1 - \frac{8}{9}\pi G T_0^2\right). \quad (25)$$

This demonstrates that in equilibrium and in the limit $T_0^2 \ll E_{\rm Planck}^2$, the thermal correction to Einstein point which follows from minimization of the free energy coincides with the result following from the conventional general relativity approach, in which the Einstein equations are solved in a self-consistent manner taking into account thermal fluctuations [14].

Although our treatment here cannot consider the dynamics of the system, we can realize that general equations for the evolution of an effective FRW Universe in contact with a fixed temperature reservoir will be different from that of Einstein. This in particular must include the evolution of the 'cosmological constant' to its equilibrium value.

5. Homogeneous but anisotropic perturbations. In standard general relativity the Einstein point for a Universe filled with radiation is also unstable against homogeneous but anisotropic perturbations of the metric of the Bianchi type IX (see for example [4]). For completeness, in this section we want to see whether the Einstein isotropic point is, on the contrary, stable in our approach.

In order to analyze the stability of the Einstein state from our emergent gravity point of view, we need to calculate the free energy of static configurations of the Bianchi IX type, which include the Einstein isotropic configuration. The general metric for these models is [15]

$$ds^2 = -N dt^2 + \sum_{n=1}^{3} a_n^2 \sigma_n^2, \quad (26)$$

$$\sigma_1 = \sin\psi d\theta - \cos\psi \sin\theta d\varphi, \quad (27)$$

$$\sigma_2 = \cos\psi d\theta + \sin\psi \sin\theta d\varphi, \quad (28)$$

$$\sigma_3 = -(d\psi + \cos\theta d\varphi). \quad (29)$$

Now, the free energy for these configurations results

$$F(a_1, a_2, a_3, N, T) = \frac{2\pi^2 a_1 a_2 a_3}{8\pi G}\left[\frac{Na_1^2}{a_2^2 a_3^2} + \frac{Na_2^2}{a_1^2 a_3^2} + \frac{Na_3^2}{a_1^2 a_2^2} - \frac{2N}{a_1^2} - \frac{2N}{a_2^2} - \frac{2N}{a_3^2} + N\Lambda - \frac{\tilde{p}}{N^3}\right]. \quad (30)$$

Again, it is not difficult to see that the Einstein point is an extremum of this free energy $a_1^2 = a_2^2 = a_3^2 = (3/2)\Lambda^{-1} = (3/2)\tilde{\rho}^{-1}$, and that it is a local minimum.

6. Discussion. The Einstein static model was introduced as a state of equilibrium for the Universe as a whole. A few years after its introduction it was pointed out that it will be unstable under perturbations. At present we know that the issue of stability of closed and static configurations is not as clear cut as it was considered, although there are still several ways to instability. For example, a radiation dominated Universe, could be driven globally unstable by a sudden decrease in pressure in the entire Universe.

Here, we have analyzed this global instability of a static Universe, in the simple case of radiation dominance, but taken a different perspective on the essence of gravity. We have considered that general relativity might be an emergent feature of an underlying quantum theory of a similar nature to that describing condensed matter system as liquid Helium. In this case (some other cases can also be imagined), it appears reasonable to consider the temperature of the Universe as something independent (at least in a first approximation) of the specific characteristics of the emergent 4-dimensional geometry. This characteristic put us outside the realm of standard general relativity.

Taking this perspective, we have calculated the free energy for static Universes of the Einstein type filled with a gas of photons at a fixed temperature. We have seen that the free energy has an extremum at the Einstein point and that this extremum correspond to a local minimum. Therefore, thermodynamically speaking the Einstein state for the Universe would be stable under the stated condition.

It is not difficult to understand why this is the case. The Eddington instability of the Einstein state is based on the following fact. The contractive tendency of matter operates stronger on short scales. Instead, the expansive tendency of the cosmological constant operates stronger on large scales. In the Einstein point these two tendencies are exactly balanced. However, if the universe is suddenly made larger, the cosmological constant effect takes over and expands further the Universe. Reciprocally, if the Universe is made smaller the matter dominates and makes the Universe to further contract. However, in the case analyzed here, a sudden expansion of the Universe will be accompanied by the introduction of more photons in the system in order to keep the temperature constant in the now larger volume. This increase on the amount of matter completely counterbalance the cosmological constant tendency making the Universe to contract back to its initial state.

We also know that the Einstein point is unstable in standard general relativity to homogeneous but anisotropic perturbations of the Bianchi type IX. Taking our point of view, we have calculated the behaviour of the Einstein point inside the general class of static Bianchi model of type IX. We have also found that the Einstein point is stable in our approach.

In summary, in the standard general relativistic view point, the radiation dominated Einstein state is unstable under global perturbation of the scale factor and also from homogeneous but anisotropic perturbations. However, from the point of view of emergent gravity these instabilities are not present.

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