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Energy spectra of developed superfluid turbulence

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Turbulence spectra in superfluids are modified by the nonlinear energy dissipation caused by the mutual friction between quantized vortices and the normal component of the liquid. We have found a new state of fully developed turbulence which occurs in some range of two Reynolds parameters characterizing the superfluid flow. This state displays both the Kolmogorov-Obukhov $\frac{5}{3}$-scaling law $E_k \propto k^{-5/3}$ and a new "3-scaling law" $E_k \propto k^{-3}$, each in a well-separated range of k.

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Superfluid consists of mutually penetrating components – viscous normal component and one or several frictionless superfluid components. This explains why different types of turbulent motion are possible depending on whether the normal and the superfluid components move together or separately. Here, we are interested in the most simple case when the dynamics of the normal component can be neglected. This occurs, for example, in the superfluid phases of ^{3}He where the normal component is so viscous that it is practically clamped to the container walls. The role of the normal component in this case is to provide the preferred heat-bath reference frame, where the normal component, and thus the heat bath, are at rest. Dissipation takes place when the vortices move with respect to this reference frame. Turbulence in such a superfluid component with the normal component at rest will be called here superfluid turbulence.

Recent experiments in ^{3}He-B [1] demonstrated that the fate of a few vortices injected into a rapidly moving superfluid depends on a dimensionless intrinsic temperature-dependent parameter q rather than on the flow velocity. At $q \sim 1$, a rather sharp transition is observed between laminar evolution of the injected vortices and a turbulent many-vortex state of the whole superfluid. This adds a new twist to the general theory of turbulence in superfluids developed by Vinen [2, 3] and others. Attempts to modify the theory in order to incorporate the new phenomenon, have been made in Refs. [4 – 6].

In this Letter we describe how the celebrated Kolmogorov-Obukhov $\frac{5}{3}$-law for the turbulent energy spectrum in normal fluid, $E_k \propto k^{-5/3}$, gets modified in the superfluid turbulence, giving rise the much steeper decrease, $E_k \propto k^{-3}$.

As a starting point we utilize a coarse-grained hydrodynamic equation for the superfluid dynamics with distributed vortices. In this equation the parameter q characterizes the friction force between the superfluid and the normal components of the liquid, which is mediated by quantized vortices. According to this equation, turbulence develops only if the friction is relatively small compared to the inertial term, i.e. when $q < 1$. Here, we will study the case of developed turbulence which must occur at $q \ll 1$.

An important feature of superfluid turbulence is that the vorticity of the superfluid component is quantized in terms of the elementary circulation quantum κ (in ^{3}He-B, $\kappa = \pi\hbar/m$ where m is the mass of ^{3}He atom). Thus, superfluid turbulence is a chaotic motion of well-determined and well-separated vortex filaments [3]. Using this as starting point we can simulate the main ingredients of classical turbulence – the chaotic dynamics of the vortex degrees of freedom of the liquid. However, to make the analogy useful for classical turbulence one must choose the regime described by equations of the hydrodynamic type valid at length-scales above the inter-vortex distance, the latter being a microscopic cut-off similar to the inter-atomic distance in conventional hydrodynamics. The coarse-grained hydrodynamic equa-

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tion for the superfluid component is obtained from the Euler equation for the superfluid velocity $\mathbf{v} \equiv \mathbf{v}_s$ after averaging over the vortex lines (see review [7]):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} + \boldsymbol{\nabla}\mu = \mathbf{D}, \quad (1)$$

where μ is the chemical potential and $\mathbf{D}$ describes the mutual friction:

$$\mathbf{D} = -\alpha'(\mathbf{v} - \mathbf{v}_n) \times \boldsymbol{\omega} + \alpha\, \hat{\boldsymbol{\omega}} \times [\boldsymbol{\omega} \times (\mathbf{v} - \mathbf{v}_n)]. \quad (2)$$

Here $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the superfluid vorticity; $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}/\omega$; $\mathbf{v}_n$ is the velocity of the normal component (which is fixed); α' and α are dimensionless parameters describing the mutual friction between superfluid and normal components of the liquid mediated by quantized vortices which transfer momenta from the superfluid to the normal subsytem. For the flow with vortices locally aligned with each other these parameters enter the reactive and dissipative forces acting on a vortex line as it moves with respect to the normal component. For vortices in fermionic systems (superfluid ^{3}He and superconductors) such forces acting on a vortex were calculated by Kopnin [8], and they were measured in ^{3}He-B over a broad temperature range by Bevan et al. [9]. Here we consider α' and α as phenomenological parameters, assuming the general case where quantized vortices are not aligned locally and thus the bare parameters are renormalized.

Further we shall work in the reference frame where $\mathbf{v}_n = 0$. In this frame, the nondissipative first term in Eq. (2) renormalizes the inertial term $\mathbf{v} \times \boldsymbol{\omega}$ in the left hand side (LHS) of Eq. (1) by the factor $1 - \alpha'$. The role of the Reynolds number in this hydrodynamics, i.e. relative magnitude of the two non-linear terms, the inertial and friction ones, is played by the velocity independent ratio of dimensionless parameters, $\mathrm{Re} = (1-\alpha')/\alpha$, which was denoted in Ref. [1] as $\mathrm{Re} = 1/q$. The role of the parameter q as the inverse Reynolds number was demonstrated in experiments of Ref. [1], where it was shown that the turbulence develops only below some critical value of q (i.e. at $q < q_c \sim 1$).

Here we are interested in the region of large Reynolds numbers, $\mathrm{Re} \gg 1$ ($q \ll 1$), where the inertial term is strongly dominating. In this region one expects well developed turbulence characterised by a Richardson-Kolmogorov-type cascade which is modified due to the non-linear dissipation. In ^{3}He-B, the range $q \ll 1$ occurs at low temperatures, where $\alpha' \ll \alpha$, $q \approx \alpha$. Then the mutual friction term in Eq. (2) can be written as

$$\mathbf{D} = q\, \boldsymbol{\omega} \times [\boldsymbol{\omega} \times \mathbf{v}]/|\omega|, \quad (3)$$

and we finally arriv at the hydrodynamic equation (1) whose LHS is the usual Euler equation with the nonlinear term which is responsible for the energy cascade in the developed hydrodynamic turbulence, while the RHS contains the nonlinear dissipation term given be Eq. (3). We will describe this cascade in the simplest possible manner, using the differential form [10, 11] of the energy transfer term in the energy budget equation (the more complicated version with second derivative [12] was used for superfluid turbulence by Vinen [6]):

$$\frac{\partial E_k}{\partial t} = -\mathcal{D}_k - \frac{\partial \varepsilon_k}{\partial k}. \quad (4)$$

Here E_k is the one-dimensional density of the turbulent kinetic energy in the k-space, defined such the the total energy density (in the physical space) E is given by

$$E \equiv \frac{1}{2}\langle |\mathbf{v}|^2 \rangle = \int dk\, E_k. \quad (5)$$

Dissipation of energy on scale k is described by the $\mathcal{D}_k$ term in the right hand side (RHS) of Eq. (4) which will be clarified later. The idea of [10] is to relate E_k and ε_k in Eq. (4) in the spirit of Kolmogorov 1941 (K41) dimensional reasoning:

$$E_k = C\varepsilon_k^{2/3} k^{-5/3}. \quad (6)$$

Here $C \simeq 1$ is the Kolmogorov dimensionless constant. In the absence of dissipation, Eq. (4) immediately produces the stationary solution $\varepsilon_k = \varepsilon$ with constant energy flux ε in the inertial interval of scales. Then Eq. (6) turns into the Kolmogorov-Obukhov $\frac{5}{3}$-law for E_k:

$$E_k = C\varepsilon^{2/3} k^{-5/3}. \quad (7)$$

The goal of this Letter is to describe possible modification of the scaling exponents in E_k due to different, than in the Navier-Stokes equation, form (3) of the dissipation term. The energy balance equation in the differential approximation (4) is just an adequate tool for this study: it is as simple as possible, but not more. More accurate integral representation for the energy transfer term gives exactly the same results for the scaling exponents, because they arise from the power counting. Clearly, approximation (4) does not accurately control numerical prefactors and the exact functional form of the possible crossover region, but these are not important for the questions we aim to study here.

Within the same level of accuracy, we can simplify the vectorial structure of the dissipation term $\mathbf{D}$ and average Eq. (3) over the directions of the vorticity $\boldsymbol{\omega}$ (at fixed direction of $\mathbf{v}$)

$$\mathbf{D} \Rightarrow \langle \mathbf{D} \rangle_{\boldsymbol{\omega}/|\omega|} = -\frac{2}{3}\, q\, |\omega| \mathbf{v} \Rightarrow -\, q\, |\omega| \mathbf{v}. \quad (8)$$

Again, we are not bothered by the numbers and therefore skipped for simplicity factor 2/3 in the last of Eqs. (8). Notice that vorticity in hydrodynamic turbulence is usually dominated by the k-eddies (i.e. motions of scale $\sim 1/k$) with the largest characteristic wave-vector $k_{\max}$.

These eddies have the smallest turnover time $\tau_{\min}$ that is of the order of their decorrelation time. One can show that the main contribution to the velocity (not vorticity) in the equation for the dissipation of the k-eddies, $\mathcal{D}_k$, with intermediate wave-vectors k, $1/R < k \ll k_{\max}$, is dominated by the k'-eddies with $k' \sim k$. Because the turnover time of these eddies $\tau_{k'} \gg \tau_{\min}$, we can think of $|\omega|$ in Eq. (8) on time intervals of interest ($\tau_{\min} \ll \tau \ll \tau_{k'}$) as self-averaging quantity because it is almost uncorrelated with the velocity $\mathbf{v}$ which can be treated as dynamical variable. In this study, this allows us to neglect in Eq. (8) the fluctuating part of $|\omega|$ and to replace $|\omega|$ by its mean value. In this approximation Eq. (8) takes very simple form:

$$\mathbf{D} = -\Gamma\,\mathbf{v}\,, \quad \Gamma \equiv q\,\omega_0\,, \quad \omega_0 \equiv \langle|\omega|\rangle. \tag{9}$$

From Eq. (9) one easily finds that

$$\mathcal{D}_k = \Gamma\,E_k, \tag{10}$$

and the balance equation (4) in the steady state finally takes the form:

$$\frac{\partial \varepsilon_k}{\partial k} = -2\,\Gamma \varepsilon_k^{2/3} k^{-5/3}, \tag{11}$$

in which for the simplicity we put $C = 1$, because in our simple approach we are not controlling numbers of the order of unity.

Let us analyze the solutions of Eq. (11) that arise in presence of a fixed energy influx $\epsilon_k = \epsilon_+$ at $k = 1/R$ into the turbulent system. Hereafter R is the outer scale of turbulence that of the order of the radius of the cryostat. In terms of dimensionless variables, $p = kR$, $f_p = \epsilon_p/\epsilon_+$, $\gamma = \Gamma R/V$ and $V = (\epsilon_+ R)^{1/3}$, we have

$$\frac{\partial f_p}{\partial p} = -2\gamma f_p^{2/3} p^{-5/3}. \tag{12}$$

The solution to this with the boundary condition $f|_{p=1} = 1$ is

$$f_p^{1/3} = \gamma p^{-2/3} + 1 - \gamma, \tag{13}$$

or, in terms of the dimensional energy spectrum,

$$E_k = \frac{V^2}{k(kR)^{2/3}} \left[1 + \frac{\gamma}{(kR)^{2/3}} - \gamma\right]^2. \tag{14}$$

Then, expressing the mean vorticity through this energy spectrum, one obtains the closed equation for the parameter γ:

$$\Gamma = \gamma\,V/R = q\omega_0(\gamma), \tag{15}$$

which manifests the existence of several different regimes of turbulence. Consider first the case when the resulting $\gamma < 1$ and Eq. (14) can be rewritten as

$$E_k = \frac{V^2\gamma^2}{R^2 k^{5/3}} \left[\frac{1}{k^{2/3}} + \frac{1}{k_{cr}^{2/3}}\right]^2, \tag{16}$$

where

$$k_{\rm cr} \equiv \frac{\gamma^{3/2}}{(1-\gamma)^{3/2} R} \tag{17}$$

is a crossover wavenumber separating two different scaling ranges. For $k \gg k_{\rm cr}$ we have the K41 scaling in which the dissipation is negligible and the energy flux is approximately constant,

$$E_k = \frac{V^2(1-\gamma)^2}{R^{2/3}k^{5/3}}. \tag{18}$$

This equation can be rewritten in the traditional form (7)

$$E_k \simeq \varepsilon_\infty^{2/3} k^{-5/3}, \tag{19}$$

in which the energy flux ε_∞ for $k > k_{\rm cr}$ due to the mutual friction can be much smaller than the energy influx, ε_+, into the turbulent system:

$$\varepsilon_\infty = \varepsilon_+(1-\gamma)^3 < \varepsilon_+. \tag{20}$$

In order to see how ε_k decreases toward large k, approaching ε_∞ at $k \sim k_{\rm cr}$ consider region $k \ll k_{\rm cr}$. In this region Eqs. (13), (14) yield:

$$\varepsilon_k = \frac{\varepsilon_+}{(kR)^2}, \quad E_k = \frac{V^2\gamma^2}{R^2\,k^3}. \tag{21}$$

Thus the rate of the energy dissipation, being proportional to E_k, decreases toward large k and becomes insignificant at $k \gg k_{\rm cr}$. In this region $\varepsilon_k \simeq \varepsilon_\infty$ and one has the K41 scaling (19).

Notice, that k^{-3} spectrum (21) corresponds to the balance between the energy flux and dissipation due to the mutual friction. This follows from the energy balance equation (11), but also can be directly understood in the r-representation. Indeed, the energy flux in the scale r can be evaluated as V_r^3/r, while the energy dissipation is ΓV_r^2. Here V_r is the characteristic velocity of eddies of the scale r, i.e. the velocity increment across the separation r. The balance yields $V_r \simeq \Gamma r$, i.e. the energy of r-eddies $V_r^2 \propto r^2$. In the k representation this gives one-dimensional energy spectrum $E_k \propto k^{-3}$, i.e. the 3-law (21).

The K41 scaling ends by a cutoff at a "microscopic" scale $1/k_*$ at which circulation in the k_*-eddy reaches the circulation quantum κ:

$$\kappa \sim \frac{v_*}{k_*} = \frac{RV(1-\gamma)}{(k_*R)^{4/3}}, \tag{22}$$

i.e.

$$(k_*R)^{4/3} \sim \mathcal{N}^2(1-\gamma), \tag{23}$$

where $\mathcal{N}^2 = RV/\kappa$ is the "quantum Reynolds number" [1]. Parameter $\mathcal{N}$ can be considered as the ratio of R to the mean inter-vortex distance. Clearly, with the classical approach to the problem we can consider only the limit $\mathcal{N} \gg 1$.

Now we can clarify the equation of self-consistency (15) for γ. Estimating ω_0 as follows

$$\omega_0^2 = \langle|\omega|\rangle^2 \simeq \langle|\omega|^2\rangle \simeq \int_{1/R}^{k_*} dk\, k^2\, E_k,$$

and using Eqs. (15) and (16) one gets

$$1 \simeq q^2 \int_{1/R}^{k_*} dk\, k^{1/3} \left[\frac{1}{k^{2/3}} + \frac{1}{k_{cr}^{2/3}}\right]^2 . \tag{24}$$

Together with Eqs. (17) and (23), that relate k_* and $k_{\rm cr}$ with γ and $\mathcal{N}$, this equation allows to find γ, k_* and $k_{\rm cr}$ in the terms of "external parameters" of the problem, q and $\mathcal{N}$. As we pointed out, the classical regime of developed turbulence corresponds to the region $q \ll 1$, $\mathcal{N} \gg 1$.

Consider first the case when the inner (quantum) scale of turbulence is well separated from the crossover scale: $k_* \gg k_{\rm cr}$. Then the main contribution to the integral in Eq. (24) comes from the region $k \gg k_{\rm cr}$, when the second term in the integral dominates. Therefore Eq. (24) gives the relationship

$$k_{\rm cr} \simeq q^{3/2} k_*, \tag{25}$$

which together with Eqs. (17) and (23) yields Eq. for γ:

$$q\mathcal{N} \simeq \gamma/(1-\gamma)^{3/2} \ . \tag{26}$$

For $q\mathcal{N} \ll 1$ the solution is $\gamma \simeq q\mathcal{N} < 1$ that gives $Rk_{\rm cr} \simeq (q\mathcal{N})^{3/2} < 1$ and:

$$E_k \simeq \frac{V^2}{k(kR)^{2/3}} \left[1 - 2\, q\mathcal{N}\left(1 - \frac{1}{(kR)^{2/3}}\right)\right], \qquad Rk_* \simeq \mathcal{N}^{3/2} \gg 1, \ \text{for } q\mathcal{N} \ll 1. \tag{27}$$

It means that for $q\mathcal{N} \ll 1$ in the entire inertial interval, $1/R < k < k_*$, one has usual Kolmogorov-Obukhov spectrum with the small, of the order of $q\mathcal{N}$, negative corrections. In other words, at $q\mathcal{N} \ll 1$ the mutual friction has a negligible effect on the statistics of turbulence.

The situation is different in the region $q\mathcal{N} \gg 1$. In this case the solution to Eq. (26) is $1-\gamma \simeq (q\mathcal{N})^{-2/3} \ll 1$ and instead of Eq. (27) one has:

$$E_k \simeq \frac{V^2}{R^2 k^{5/3}} \left[\frac{1}{k^{2/3}} + \frac{1}{k_{\rm cr}^{2/3}}\right]^2, \ \text{for } q\mathcal{N} \gg 1, \qquad k_{\rm cr} R \simeq q\mathcal{N}, \quad k_* R \simeq \mathcal{N}/q^{1/2} \gg k_{\rm cr} R. \tag{28}$$

One sees that the pumping and the crossover scales are well separated if $q\mathcal{N} \gg 1$. Under this condition, the quantum cutoff scale is also well separated.

Up to now we assumed that the second term in $[\dots]$ in the integral (24) is dominant. In the opposite case instead of Eqs. (25) and (26) one has:

$$k_* R \simeq \exp\left(\frac{1}{q^2}\right), \quad k_{\rm cr} R \simeq \mathcal{N}^3 \exp\left(-\frac{2}{q^2}\right). \tag{29}$$

Taking into account that we are considering solutions in which the first term in $[\dots]$ in the integral (24) is dominant we have to take $k_{\rm cr} > k_*$ which gives $\mathcal{N} > \exp(1/q^2)$. Therefore the range of parameters where the two-cascade regime (28) occurs is

$$1/q < \mathcal{N} < \exp(1/q^2). \tag{30}$$

An important feature of this solution is that both the energy spectrum E_k and the spectrum of the flow dissipation ε_k are concentrated at the largest length scale R, whereas the dissipation is mediated by the vorticity ω_0 concentrated at the smallest (microscopic) scale $1/k_*$. Therefore the energy balance between the Kolmogorov cascade and the energy dissipation must occur already for the largest eddies. This gives the condition [5]

$$\frac{V^3}{R} = \Gamma V^2. \tag{31}$$

This means that in this turbulent state the mean vorticity is $\omega_0 = \Gamma/q = U/qR$. If $q\mathcal{N} \gg 1$ one has γ close to 1 and thus our double-cascade solution (28) satisfies the large-scale balance (31).

At $q\mathcal{N} \sim 1$ one has $k_{\rm cr} = 1/R$, i.e. the region of the k^{-3} spectrum shrinks. At $q\mathcal{N} \ll 1$ the parameter γ deviates from unity, $\gamma \simeq q\mathcal{N}$. Here two scenaria are possible. In the first one we have solution (27) in which the mutual friction is unessential and thus is unable to compensate the Kolmogorov cascade. When the intervortex distance scale is reached the Kolmogorov energy cascade is then transformed to the Kelvin wave cascade [3] for the isolated vortices. In the second scenario suggested in Refs. [5] and [13], at $q\mathcal{N} \ll 1$ the turbulent state is completely reconstructed and the so called Vinen state emerges. This state introdiced by Vinen [14] and then by Schwarz [15] contains a single scale $r = \kappa/V$ and thus no cascade.

Another interesting case to consider is $\gamma > 1$ when Eq. (14) for E_k can be rewritten as follows:

$$E_k = \frac{V^2\gamma^2}{R^2 k^{5/3}} \left[\frac{1}{k^{2/3}} - \frac{1}{\tilde{k}_{\rm cr}^{2/3}}\right]^2, \tag{32}$$

where

$$\tilde{k}_{\rm cr} = \frac{\gamma^{3/2}}{(\gamma-1)^{3/2} R}. \tag{33}$$

Here the Kolmogorov cascade stops already at the scale $\tilde{k}_{\rm cr}$, and the equation for γ reads:

$$1 \simeq q\sqrt{\frac{3}{2}\ln\frac{\gamma}{(\gamma-1)}}. \quad (34)$$

The solution satisfying the large-scale energy balance $\gamma \simeq 1$ is

$$\gamma - 1 \simeq \exp(-2/3q^2). \quad (35)$$

This solution is self-consistent and does not require the microscopic scale cut-off if the circulation at the scale $\tilde{k}_{\rm cr}$ is big enough, i.e. $v_{\rm cr}/\tilde{k}_{\rm cr} \gg \kappa$. This occurs, however, at very large counterflow $\mathcal{N} \gg \exp(1/q^2)$. However, we think that this solution is unstable. Indeed, let us consider a perturbation of this spectrum in a form of cutting off its tail. This will lead to a reduction in the dissipation rate so that the subsequent evolution will build a K41 constant-flux tail rather than restore the k^{-3} scaling. The K41 tail will strengthen until its contribution to the friction will restore the energy balance. Thus, the resulting new steady state will have the K41 part, i.e. will be of the first kind.

The summary of the different regimes (without possible Vinen state) is shown in the Table.

Scaling range boundaries in the cases of weak, intermediate and strong pumping

Intensity, $\mathcal{N}$	Crossover, $k_{cr}R$	Quantum cutoff, k_*R
$1 < \mathcal{N} < 1/q$	None, $\frac{5}{3}$-scaling	$\mathcal{N}^{3/2}$
$1 < 1/q < \mathcal{N} < e^{1/q^2}$	$q\mathcal{N}$	$\mathcal{N}/\sqrt{q}$
$1 < e^{1/q^2} < \mathcal{N}$	None, 3-scaling	e^{1/q^2}

Now let us compare our results for the superfluid turbulence with earlier works. In Refs. [5] and [13] the effect of the mutual friction on the high momentum tail was overestimated, which led to the incorrect result for the spectrum of dissipation at high momenta. However, some general features suggested in Ref. [5] remain the same. In particular, there are different regimes of the superfluid turbulence. The transition line between two turbulent regimes, $q\mathcal{N} \simeq 1$, was correctly determined in Refs. [5, 13], as well as the vorticity in the regime of cascade, $\omega_0 = V/qR$. Recently Vinen [6] used a diffusion equation model which is similar in spirit to the model used in our Letter and, perhaps, even better because in properly accounts not only for the cascade states but also for the thermodynamic equilibria. However, this equation is harder to solve analytically and Vinen used numerical simulations. His qualitative and numerical results are consistent with our analytic solution.

In conclusion, we discussed the spectrum of the superfluid turbulence governed by the nonlinear energy dissipation due to the mutual friction between the vortices and the normal component of the liquid, which remains at rest. We found that in agreement with Refs. [1, 5], the flow states are determined by two dimensionless parameters: the velocity-independent Reynolds number $\mathrm{Re} = 1/q$ which separates the laminar and turbulent states; and the quantum velocity-dependent parameter $\mathcal{N} = \sqrt{VR/\kappa}$ which contains the quantum of circulation around the quantum vortices κ and which determines the transition, or crossover, between different regimes of superfluid turbulence. In some region of the $(q, \mathcal{N})$ plane, we found the turbulent state with a well defined Richardson-type cascade. This state displays both the Kolmogorov-Obukhov $\frac{5}{3}$-scaling law $E_k \propto k^{-5/3}$ and the new "3-scaling law" $E_k \propto k^{-3}$, each in a well separated range of k. Possible connection of the phase diagram of the flow states to experimental observations is discussed in Ref. [16].

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