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# On Kelvin–Helmholtz instability in superfluids

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The Kelvin-Helmholtz instability in superfluids is discussed, based on the first experimental observation of such instability at the interface between superfluid $^3$He-A and superfluid $^3$He-B (R. Blaauwgeers, V. B. Eltsov, G. Eska et al., cond-mat/0111343). We discuss why the Kelvin–Helmholtz criterion, the Landau critical velocity for nucleation of ripplons, and the free energy consideration all give different values for the instability treshold.

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**1. Classical Kelvin-Helmholtz (KH) instability.** KH instability belongs to a broad class of interfacial instabilities in liquids, gases, plasma, etc. [1]. It refers to the dynamic instability of the interface of the discontinuous flow, and may be defined as the instability of the vortex sheet. Many natural phenomena have been attributed to this instability. Most familiar of them are generation by wind of waves in the water, whose Helmholtz instability [2] was first analyzed by Kelvin [3], and flapping of sails and flags analyzed by Rayleigh [4] (see recent experiments in [5]).

Many of the leading ideas in the theory of instability were originally inspired by considerations about inviscid flows. The corrugation instability of the interface between two ideal liquids sliding with along each other was first investigated by Lord Kelvin [3, 6]. The critical relative velocity $|v_1 - v_2|$ for the onset of corrugation instability is given by

$$\frac{1}{2}\frac{\rho_1\rho_2}{\rho_1+\rho_2}(v_1-v_2)^2=\sqrt{\sigma F}\ . \qquad (1)$$

Here $\sigma$ is surface tension of the interface between two liquids; $\rho_1$ and $\rho_2$ are their mass densities; and $F$ is related to the external field stabilizing the position of the interface: typically it is the gravitational field

$$F=g(\rho_1-\rho_2)\ . \qquad (2)$$

The surface mode (ripplon) which is excited first has the wave vector

$$k_0=\sqrt{F/\sigma}\ , \qquad (3)$$

and frequency

$$\omega_0=k_0\frac{\rho_1 v_1+\rho_2 v_2}{\rho_1+\rho_2}\ . \qquad (4)$$

The excited ripplon propagates along the interface with the phase and group velocity: $v_{\rm ripplon} = (\rho_1 v_1 + \rho_2 v_2)/(\rho_1+\rho_2)$.

However, among the ordinary liquids one cannot find an ideal one. That is why in ordinary liquids and gases it is not easy to correlate theory with experiment. In particular, this is because one cannot properly prepare the initial state – the planar vortex sheet is never in equilibrium in a viscous fluid: it is not the solution of the hydrodynamic equations if viscosity is finite. That is why it is not so apparent whether one can properly discuss its 'instability'.

Superfluids are the only proper ideal objects where these ideas can be implemented without reservations, and where the criterion of instability does not contain viscosity. Recently the first experiment has been performed in superfluids, where the nondissipative initial state was well determined, and the well defined threshold has been reported [7]. The initial state is the nondissipative vortex sheet separating two sliding superfluids. One of the superfluids ($^3$He-A) performs the solid-body like rotation together with the vessel, while in the other one ($^3$He-B) the superfluid component is in the so-called Landau state, i.e. it is vortex-free and thus is stationary in the inertial frame. The threshold of the Kelvin-Helmholtz type instability has been marked by formation of vortices in the vortex-free stationary superfluid: this initially stationary superfluid starts to spin-up by the neighboring rotating superfluid.

**2. KH instability in superfluids at low $T$.** The extension of the consideration of classical KH instability to superfluids adds some new physics. First of all, it is now the two-fluid hydrodynamics with superfluid and normal components which must be incorporated. Let us first consider the limit case of low $T$, where the fraction of the normal component is negligibly small, and thus the complication of the two-fluid hydrodynamics is avoided. In this case one may guess that the classical result (1) obtained for the ideal inviscid liquids is applicable for superfluids too, and the only difference is that the role of the gravity is played by the applied gradient of magnetic field $H$, which stabilizes the position of the interface between $^3$He-A and $^3$He-B in the experiment [7]:

$$F = (1/2)(\chi_A(T) - \chi_B(T))\nabla(H^2). \qquad (5)$$

Here $\chi_A$ and $\chi_B$ are temperature dependent magnetic susceptibilities of the A and B phases.

However, this is not the whole story. The instability will start earlier, if one takes into account that there is a preferred reference frame. It can be the frame of container, the frame of the crystal in superconductors, or even the frame where the inhomogeneity of magnetic field $H$ is stationary. The energy of the excitations of the surface, ripplons, can become negative in this reference frame, and the surface modes will be excited, before the onset of the classical KH instability.

Let us consider this phenomenon. We repeat the same derivation as in case of classical KH instability, assuming the same boundary conditions, but with one important modification: in the process of the dynamics of the interface one must add the friction force arising when the interface is moving with respect to the container wall. In the frame of the container, which coincides with the frame of the stable position of the interface, the friction force between the interface and container is

$$F_{\rm friction} = -\Gamma \partial_t \zeta \ , \qquad (6)$$

where $\zeta(x,t)$ is perturbation of the position of the interface:

$$z = z_0 + \zeta(x,t) \ , \ \zeta(x,t) = a \sin(kx - \omega t) \ . \qquad (7)$$

We assume that the velocities $v_1$ and $v_2$ are both along the axis $x$; the container walls are parallel to the $(x,z)$-plane; and the interface is parallel to the $(x,y)$-plane.

The friction force in Eq.(6) violates the Galilean invariance in $x$-direction, which reflects the existence of the preferred reference frame – the frame of container. This symmetry breaking is the main reason of the essential modification of the KH instability. The parameter $\Gamma$ in the friction force has been calculated for the case when the interaction between the interface and container is transferred by the normal component of the liquid due to Andreev scattering of ballistic quasiparticles by the interface [8]. The friction modifies the classical spectrum of surface modes:

$$\rho_1\left(\frac{\omega}{k} - v_1\right)^2 + \rho_2\left(\frac{\omega}{k} - v_2\right)^2 = \frac{F + k^2\sigma}{k} - i\Gamma\frac{\omega}{k} \ , \qquad (8)$$

or

$$\frac{\omega}{k} = \frac{\rho_1 v_1 + \rho_2 v_2}{\rho_1 + \rho_2} \pm$$
$$\pm \frac{1}{\sqrt{\rho_1 + \rho_2}}\sqrt{\frac{F + k^2\sigma}{k} - i\Gamma\frac{\omega}{k} - \frac{\rho_1\rho_2}{\rho_1 + \rho_2}(v_1 - v_2)^2}, \qquad (9)$$

where $v_1$ and $v_2$ are the velocities of superfluid components of the liquids with respect to the container walls.

For $\Gamma = 0$ the spectrum of ripplons acquires the imaginary part, $\mathrm{Im}\,\omega(k) \neq 0$, at the classical treshold value in Eqs.(1) and (3). However, the frame-fixing parameter $\Gamma$ leads to essentially different result: The imaginary part of frequency becomes positive $\mathrm{Im}\,\omega(k) > 0$ first for ripplons with the same value of the wave vector, as in Eq.(3), but the ripplon frequency is now $\omega = 0$ and its group velocity is $v_{group} = d\omega/dk = 0$. The critical ripplon is stationary in the reference frame of the container, as a result the onset of instability is given by

$$\frac{1}{2}\rho_1 v_1^2 + \frac{1}{2}\rho_2 v_2^2 = \sqrt{\sigma F} \ . \qquad (10)$$

This criterion does not depend on relative velocities of superfluids, but is determined by velocities of each of the two superfluids with respect to the container (or to the remnant normal component). Thus the instability can occur even if two liquids have equal densities, $\rho_1 = \rho_2$, and move with the same velocity, $v_1 = v_2$. This situation is very similar to the phenomenon of flapping flag in wind, discussed by Rayleigh in terms of the KH instability – the instability of the passive deformable membrane between two distinct parallel streams having the same density and the same velocity (see latest experiments in Ref. [5]). In our case the role of the flag is played by the interface, while the role of the flagpole which pins the flag (and thus breaks the Galilean invariance) is played by the container wall.

Note that in the limit of the vanishing pinning parameter $\Gamma \to 0$ the Eq.(10) does not coincide with the classical equation (1) obtained when there is no pinning, i.e. when $\Gamma$ is exactly zero. Such difference between the limit and exact cases is known in many area of physics. In classical hydrodynamics the normal mode of inviscid theory may not be the limit of a normal mode of viscous theory [9]. Below we discuss this difference for the case of KH instability in superfluids.

**3. KH instability and modified Landau criterion.** Let us first compare both results, with no pinning ($\Gamma = 0$) and for vanishing pinning ($\Gamma \to 0$), with the Landau criterion. According to Landau, a quasiparticle is created by the moving superfluid if its velocity with respect to the container wall (or with respect to the body moving in superfluid) exceeds

$$v_{\rm Landau} = \min \frac{E(p)}{p} \ . \qquad (11)$$

Let us recall that the energy $E(p)$ here is the quasiparticle energy in the reference frame moving with the superfluid vacuum. In our case there are two superfluids moving with different velocities. That is why there is no unique superfluid comoving frame, where $E(p)$ can be uniquely determined. Such frame appears only in particular cases, when either $v_1 = v_2$, or if instead of the interface one considers the free surface of a single liquid (i.e. if $\rho_2 = 0$). In these particular cases the Landau criterion in the form of Eq.(11) must work. The energy spectrum of the ripplons on the interface between two stationary fluids (or on the surface of the single liquid) is given by Eq.(9) with $v_1 = v_2 = \Gamma = 0$:

$$\frac{\omega^2(k)}{k^2} = \frac{1}{\rho_1 + \rho_2} \frac{F + k^2\sigma}{k} \ . \qquad (12)$$

This gives the following Landau critical velocity:

$$v^2_{\rm Landau} = \min \frac{\omega^2(k)}{k^2} = \frac{2}{\rho_1 + \rho_2}\sqrt{F\sigma} \ . \qquad (13)$$

This coincides with the Eq.(10) if $v_1 = v_2$, or if $\rho_2 = 0$. But this does not coincide with the classical KH result: the latter is obtained at $\Gamma = 0$ when the interaction with the reference frame of the container is lost, and thus the Landau criterion is not applicable.

In the general case, when neither of the two conditions ($v_1 = v_2$, or $\rho_2 = 0$) fulfils, the Landau criterion must be reformulated: the instability occurs, when the frequency of the surface mode in the frame of the container crosses zero for the first time: $\omega(k; v_1, v_2) = 0$. Inspection of Eq.(9) with $\Gamma = 0$ shows that for $k = k_0$ the spectrum with negative square root touches zero just when the treshold (10) is reached. Thus the Landau criterion in its general formulation coincides with the criterion of instability obtained for the case of nonzero friction force. As distinct from the Landau criterion in the form of (11) valid for a single superfluid velocity, where it is enough to know the ripplon spectrum in the frame where the superfluid (*s*) is (are) at rest, in the general case one must calculate the ripplon spectrum $\omega(k; v_1, v_2)$ for the relatively moving superfluids.

**4. Matching zero-pinning and vanishing-pinning regimes.** The difference in the result for onset of KH instability in the two regimes – with $\Gamma = 0$ and with $\Gamma \neq 0$ – disappears only in the case when two superfluids move in such a way that in the reference frame of container the combination $\rho_1 v_1 + \rho_2 v_2 = 0$. In this arrangement, according to Eq.(4), the frequency of the ripplon created by classical KH instability is zero in the container frame. Thus at this special condition the two criteria, zero pinning (1) and vanishing pinning (10), must coincide; and they really do.

If $\rho_1 v_1 + \rho_2 v_2 \neq 0$, the crossover between the zero pinning regime and the regime of small pinning occurs by varying the observation time. Let us consider this on the example of the experimental set-up [7] with the vortex-free B-phase and the vortex-full A-phase in the rotating vessel: In the container frame one has $\mathbf{v}_1 = \mathbf{v}_{sA} = 0$, $\mathbf{v}_2 = \mathbf{v}_{sB} = -\Omega \times \mathbf{r}$; the densities of two liquids, $^3$He-A and $^3$He-B, are the same with high accuracy: $\rho_A = \rho_B = \rho$. In the non-zero pinning regime the instability occurs at the boundary of the vessel, where the velocity of the $^3$He-B is maximal, when this maximal velocity reaches the value:

$$v_c^2 = \frac{2}{\rho}\sqrt{F\sigma} = \frac{1}{2}v^2_{\rm KH} = 2v^2_{\rm Landau} \, . \qquad (14)$$

This velocity is by $\sqrt{2}$ smaller than that given by classical KH equation (1) for the zero-pinning regime. On the other hand it is by $\sqrt{2}$ larger than the Landau criterion in the form of Eq.(11), but coincides with Landau criterion properly formulated for two superfluids.

From Eq.(8) it follows that slightly above this treshold the increment of the exponential growth of the interface perturbation is

$$\mathrm{Im}\ \omega(k_0) = \frac{\Gamma k_0}{2\rho}\left(\frac{v_{sB}}{v_c} - 1\right), \quad \text{at} \quad v_{sB} - v_c \ll v_c \, . \qquad (15)$$

In the vanishing pinning limit $\Gamma \to 0$ the increment becomes small and the discussed instability of the surface has no time for development if the observation time is short enough. It will start only at higher velocity of rotation when the classical treshold of KH instability, $v_{KH}$ in Eq.(1), is reached. Thus, experimental results in this limit would depend on the observation time – the time one waits for the interface to be coupled to the laboratory frame and for the instability to develop. For sufficiently short time one will measure the classical KH criterion (1), while for the sufficiently long observation time the modified KH criterion (14) will be observed.

**5. Thermodynamic instability.** Let us now consider the case of nonzero $T$, when each of the two liquids contain superfluid and normal components. In this case the analysis requires the $2\times2$-fluid hydrodynamics. This appears to be rather complicated problem, taking into account that in some cases the additional degrees of freedom related to the interface itself must be also added. The two-fluid hydrodynamics has been used for investigation of the instability of the free surface of superfluid $^4$He by the relative motion of the normal component of the liquid with respect to the superfluid one [10]. We avoid all these complications assuming that the viscosity of the normal components of both liquids is high, as it actually happens in superfluid $^3$He. In this high-viscosity limit we can neglect the dynamics of the normal components, which is thus fixed by the container walls. Then the problem is reduced to the problem of the thermodynamic instability of the superflow in the presence of the interface.

We start with the following initial non-dissipative state corresponding to the thermal equilibrium in the presence of the interface and superflows. In thermal equilibrium the normal component must be at rest in the container frame, $\mathbf{v}_{n1} = \mathbf{v}_{n2} = 0$, while the superfluids can move along the interface with velocities $\mathbf{v}_{s1}$ and $\mathbf{v}_{s2}$ (here the velocities are in the frame of the container).

The onset of instability can be found from free-energy consideration: When the free energy of static perturbations of the interface becomes negative in the frame of the container, the initial state becomes thermodynamically unstable. The free-energy functional for the perturbations of the interface in the reference frame of the container is determined by 'gravity', surface tension, and perturbations $\tilde{\mathbf{v}}_{s1} = \nabla\Phi_1$ and $\tilde{\mathbf{v}}_{s2} = \nabla\Phi_2$ of the velocity field caused by deformation of the interface:

$$\mathcal{F}\{\zeta\} = \frac{1}{2}\int dx \Bigg( F\zeta^2 + \sigma(\partial_x\zeta)^2 + \\ + \int_{-\infty}^{\zeta} dz \rho_{s1ik}\tilde{v}^i_{s1}\tilde{v}^k_{s1} + \int_{\zeta}^{\infty} dz \rho_{s2ik}\tilde{v}^i_{s2}\tilde{v}^k_{s2}\Bigg). \quad (16)$$

For generality we discuss anisotropic superfluids, whose superfluid densities are tensors (this occurs in $^3$He-A). The velocity perturbation fields $\tilde{v}_{sk} = \nabla\Phi_k$, obeying the continuity equations $\partial_i(\rho^{ik}_s\tilde{v}_{sk}) = 0$, have the following form:

$$\Phi_1(x, z<0) = A_1 e^{k_1 z}\cos kx, \\ \Phi_2(x, z>0) = A_2 e^{-k_2 z}\cos kx, \quad (17)$$

$$\rho_{s1z}k_1^2 = \rho_{s1x}k^2 \ , \ \rho_{s2z}k_2^2 = \rho_{s2x}k^2. \quad (18)$$

The connection between the deformation of the surface, $\zeta(x) = a\sin kx$, and the velocity perturbations follow from the boundary conditions.

Because of large viscosity of the normal component it is clamped by the boundaries of the vessel. Then from the requirement that the mass and the heat currents are conserved across the wall, one obtains that the superfluid velocity in the direction normal to the wall must be zero: $\mathbf{v}_{s1}\cdot\mathbf{n} = \mathbf{v}_{s2}\cdot\mathbf{n} = 0$. This gives the following boundary conditions for perturbations:

$$\partial_z\Phi_1 = v_{s1}\partial_x\zeta, \quad \partial_z\Phi_2 = v_{s2}\partial_x\zeta \ . \quad (19)$$

Substituting this to the free-energy functional (16), one obtains the quadratic form of the free energy of the surface modes

$$\mathcal{F}\{\zeta\} = \frac{1}{2}\sum_k |\zeta_k|^2 \times \\ \times \left(F + k^2\sigma - k\left(\sqrt{\rho_{sx1}\rho_{sz1}}v^2_{s1} + \sqrt{\rho_{sx2}\rho_{sz2}}v^2_{s2}\right)\right) \quad (20)$$

This energy becomes negative for the first time for the mode with $k_0 = (F/\sigma)^{1/2}$ when

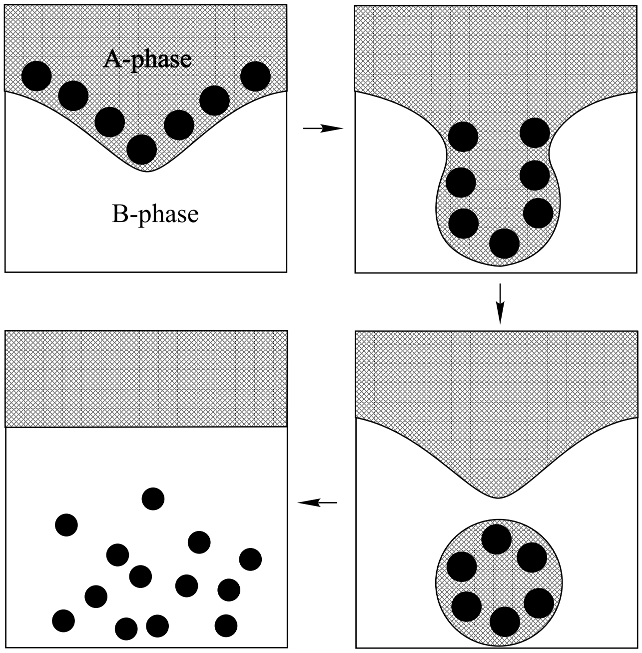
$$\frac{1}{2}\left(\sqrt{\rho_{sx1}\rho_{sz1}}v^2_{s1} + \sqrt{\rho_{sx2}\rho_{sz2}}v^2_{s2}\right) = \sqrt{\sigma F} \ . \quad (21)$$

This is the criterion (10) for the non-zero pinning regime extended to finite temperatures. Eq.(21) transforms to Eq.(10) when $T \to 0$: The normal components of the liquids disappear and one has $\rho_{sx1} = \rho_{sz1} = \rho_1$ and $\rho_{sx2} = \rho_{sz2} = \rho_2$.

**6. Nonlinear stage of instability.** Eq.(21) is in excelent agreement with the onset of the surface instability measured in experiment [7]. The onset of instability is marked by the appearance of the vortex lines in $^3$He-B which are monitored in NMR measurements. This demonstrates that vortices appear in the nonlinear stage of this KH instability.

The precise mechanism of the vortex formation is not yet known. One may guess that the A-phase vorticity is pushed by the Magnus force towards the vortex-free B-phase region [11]. When the potential well for vortices is formed by the corrugation of the interface (see Figure), the vortices are pushed there and enhance further the growth of the potential well, until it forms the droplet of the A-phase filled by vorticity. The vortex-full droplet propagates to the bulk B-phase where it relaxes to the singular vortex lines of $^3$He-B.

Under the conditions of the experiment the nucleation of vortices leads to decrease of the B-phase velocity below the instability threshold, and the vortex formation is stopped. That is why one may expect that the vortex-full droplet is nucleated during the development of the instability from a single seed. The size of the seed is about one-half of the wavelength $\lambda_0 = 2\pi/k_0$ of the perturbation. The number of the created vortices is




Possible scenario of vortex formation by Kelvin-Helmholtz instability of the AB interface

found from the circulation of superfluid velocity carried by the piece of the vortex sheet of size $\lambda_0/2$, which is determined by the jump of superfluid velocity across the sheet: $\kappa = |\mathbf{v}_{sB} - \mathbf{v}_{sA}|\lambda_0/2$. Dividing this by the circulation quantum $\kappa_0$ of the created B-phase vortices one obtains the number of vortices produced as the result of the growth of one segment of the perturbation:

$$N = \frac{\kappa}{\kappa_0} \sim \frac{v_c \lambda_0}{2\kappa_0} \ . \tag{22}$$

It is about 10 vortices per event under condition of the experiment, which is in a good agreement with the measured number of vortices created per event [7]. This is in favour of the droplet mechanism of vortex formation.

Probably, the experiments on KH instability in superluids will allow to solve the similar problem of the non-linear stage of instability in ordinary liquids (see, for example, Ref. [12]).

The vortex formation by surface instability is rather generic phenomenon. This mechanism has been discussed for vortex formation in the laser manipulated Bose gases [13, 14]. It can be applicable to different kinds of interfaces, and under very different physical conditions. In particular, vortices can be generated at the second order phase boundary between the normal and the superfluid phases [15]. Such an interface naturally appears at the rapid phase transition into the superfluid state [16]. The instability of the free surface of superfluid under the relative flow of the normal and superfluid components of the same liquid has been recently reexamined by Korshunov [17]. He also obtained two criteria of instability: for zero and nonzero values of the viscosity of the normal component of the liquid.

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