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Contribution of the $\pi^0\gamma$ and $\eta\gamma$ intermediate states to the vacuum polarization and the muon anomalous magnetic moment

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Using new experimental data, we calculate the contribution to the anomalous magnetic moment of the muon from the $\pi^0\gamma$ and $\eta\gamma$ intermediate states in the vacuum polarization with high precision taking into account the correction for using the trapezoidal rule: $a_\mu(\pi^0\gamma) + a_\mu(\eta\gamma) = (53.1 \pm 1.5) \cdot 10^{-11}$. We also find the small contribution from $e^+e^-\pi^0$, $e^+e^-\eta$ and $\mu^+\mu^-\pi^0$ intermediate states equal to $0.5 \cdot 10^{-11}$.

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New experimental data [1 – 3] allows to calculate contribution to the anomalous magnetic moment of the muon $a_\mu \equiv (g_\mu - 2)/2$ from the $\pi^0\gamma$ and $\eta\gamma$ intermediate states in the vacuum polarization with high precision. We have also found the contribution from $e^+e^-\pi^0$, $e^+e^-\eta$ and $\mu^+\mu^-\pi^0$ intermediate states.

The contribution to a_μ from the arbitrary intermediate state X (hadrons, hadrons+γ, etc.) in the vacuum polarization can be obtained via the dispersion integral

$$a_\mu = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int \frac{ds}{s^2} K(s) R(s), \qquad (1)$$

$$R(s) \equiv \frac{\sigma(e^+e^- \to X)}{\sigma(e^+e^- \to \mu^+\mu^-)}, \ \sigma(e^+e^- \to \mu^+\mu^-) \equiv \frac{4\pi\alpha^2}{3s},$$

$$K(s > 4m_\mu^2) = \frac{3s}{m_\mu^2}\Big\{x^2(1 - \frac{x^2}{2}) + (1+x)^2(1 + \frac{1}{x^2})\times$$

$$\times\Big[\ln(1+x) - x + \frac{x^2}{2}\Big] + \frac{1+x}{1-x}x^2\ln(x)\Big\} =$$

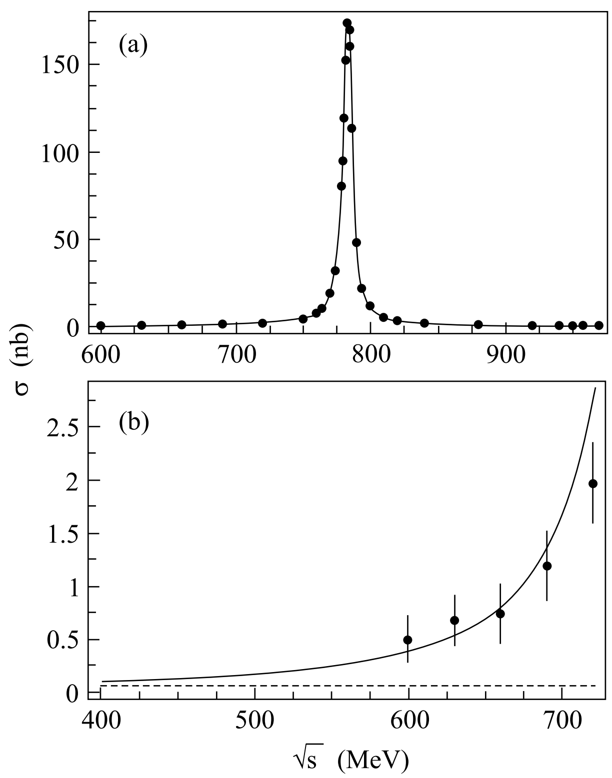
$$= \frac{3}{a^3}\Big(16(a-2)\ln\frac{a}{4} - 2a(8-a) -$$

$$-8(a^2 - 8a + 8)\frac{\operatorname{arctanh}(\sqrt{1-a})}{\sqrt{1-a}}\Big),$$

$$x = \frac{1 - \sqrt{1 - 4m_\mu{}^2/s}}{1 + \sqrt{1 - 4m_\mu{}^2/s}}, \qquad a = \frac{4m_\mu^2}{s}.$$

$$K(s < 4m_\mu^2) = \frac{3}{a^3}\Big(16(a-2)\ln\frac{a}{4} - 2a(8-a) -$$

$$-8(a^2 - 8a + 8)\frac{\arctan(\sqrt{a-1})}{\sqrt{a-1}}\Big).$$



a) Plot of the dependence $\sigma(e^+e^- \to \pi^0\gamma)$ in nb upon $\sqrt{s}$ in MeV (SND experimental data and fit). b) Comparison of the theoretical formulas for $\sigma(e^+e^- \to \pi^0\gamma)$. Eq. (6) is shown with the solid line, point-like model prediction is shown with the dashed line

Evaluating integral (1) with the trapezoidal rule for the experimental data from SND [1, 2], see Figure a, we found the contribution of $\pi^0\gamma$:

$$a_\mu^{trap}(\pi^0\gamma) = (46.2 \pm 0.6 \pm 1.3) \cdot 10^{-11}, \qquad (2)$$
$$600 \text{ MeV} < \sqrt{s} < 1039 \text{ MeV}.$$

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The first error is statistical, the second is systematic. Note that the contribution from the ϕ region ($970\ \text{MeV} < \sqrt{s} < 1039\ \text{MeV}$) is $0.7 \cdot 10^{-11}$.

At our level of precision it is necessary to take into account the error of trapezoidal rule. The point is that using trapezoidal rule for an exact value $R(s)$ we calculate not the integral, but the sum. So there is a problem to remove error of trapezoidal rule. In our case we use three SND fits for $R(s)$ from [1] based on vector dominance model [4], see Figure, for experimental data in the energy region $600\ \text{MeV} < \sqrt{s} < 970\ \text{MeV}$.

We construct new small quantity which is the correction for the error of trapezoidal rule

$$\Delta a_\mu(\pi^0\gamma, E_1, E_N) = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_1}^{s_N} \frac{ds}{s^2} K(s)R(s) - \\ - \sum_{i=1}^{N-1} \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \frac{s_{i+1} - s_i}{2} \times \\ \times \left[\frac{K(s_{i+1})R(s_{i+1})}{s_{i+1}^2} + \frac{K(s_i)R(s_i)}{s_i^2}\right], \quad (3)$$

where E_i are experimental energies, $E_1 = 600\,\text{MeV}$, $E_N = 970\,\text{MeV}$, $s_i = E_i^2$, $R(s_i)$ are the SND fits values. We get

$$\Delta a_\mu(\pi^0\gamma, 600\ \text{MeV}, 970\ \text{MeV}, fit) = -1.5 \cdot 10^{-11}. \quad (4)$$

This result varies less than at 1% for different fits. Note that statistical error is also negligible ($0.01 \cdot 10^{-11}$). So we neglect errors in (4), add it to (2) and get the result

$$a_\mu(\pi^0\gamma) = (44.7 \pm 0.6 \pm 1.3) \cdot 10^{-11}, \\ 600\ \text{MeV} < \sqrt{s} < 1039\ \text{MeV}. \quad (5)$$

The point is that values of both two items in (3), calculated using fit, have non-negligible model error. The contribution $a_\mu(\pi^0\gamma, 600\ \text{MeV}, 970\ \text{MeV})$ calculated using SND fits differs at $\sim 0.4 \cdot 10^{-11}$ from fit to fit. But the difference $\Delta a_\mu(\pi^0\gamma, E_1, E_N)$ has negligible model error due to the inequality $\Delta a_\mu(\pi^0\gamma, E_1, E_N) \ll \ll a_\mu(\pi^0\gamma, E_1, E_N)$, though relative model error is of the same order.

For the energy region $\sqrt{s} < 600\,\text{MeV}$ we use theoretical formula for the cross-section:

$$\sigma(e^+e^- \to \pi^0\gamma) = \frac{8\alpha f^2}{3}\left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 \frac{1}{\left(1 - s/m_\omega^2\right)^2}, \quad (6)$$

where $f^2 = (\pi/m_{\pi^0}^3)\Gamma_{\pi^0\to\gamma\gamma} \cong 10^{-11}/\text{MeV}^2$ according to [5]. Eq. (6) has been written in the approximation

$$\Gamma_\rho = \Gamma_\omega = 0, \quad m_\rho - m_\omega = 0. \quad (7)$$

The $\gamma* \to \pi^0\gamma$ amplitude is normalized on the $\pi^0 \to \gamma\gamma$ one at $s = 0$. The result is

$$a_\mu(\pi^0\gamma) = 1.3 \cdot 10^{-11}, \quad \sqrt{s} < 600\ \text{MeV}. \quad (8)$$

Note that the region $\sqrt{s} < 2m_\mu$ gives the negligible contribution $2 \cdot 10^{-13}$.

We neglect the small errors dealing with the experimental error in the width $\Gamma_{\pi^0\to\gamma\gamma}$ (7%) and the approximation (7) (1.5%).

The Eq. (6) agrees with the data in the energy region $\sqrt{s} < 700\,\text{MeV}$, at higher energies the approximation (7) does not work carefully, see Figure b.

If we use the point-like model, as in [6], we will get Eq. (6) without factor $\left(1 - s/m_\omega^2\right)^{-2}$. This formula predicts the contribution from low energies several times less than (8), see also Figure b.

Treating the data from CMD-2 and SND [2, 3] in the same way and combining the results, we get contribution of $\eta\gamma$:

$$a_\mu(\eta\gamma) = (7.0 \pm 0.2 \pm 0.2) \cdot 10^{-11}, \\ 720\ \text{MeV} < \sqrt{s} < 1040\ \text{MeV}. \quad (9)$$

Note that correction for the trapezoidal rule result is $-0.3 \cdot 10^{-11}$.

According to the quark model (and the model of vector dominance also), the energy region $\sqrt{s} < 720\ \text{MeV}$ is dominated by the ρ-resonance, hence $\sigma(e^+e^- \to \eta\gamma) \cong \cong \sigma(e^+e^- \to \rho \to \eta\gamma)$. So we change Eq. (6) according to this fact, take into account the ρ width and get the small contribution:

$$a_\mu(\eta\gamma) = 0.1 \cdot 10^{-11}, \quad \sqrt{s} < 720\ \text{MeV}, \quad (10)$$

Summing (5), (8), (9) and (10), we can write

$$a_\mu(\pi^0\gamma) + a_\mu(\eta\gamma) = (53.1 \pm 0.6 \pm 1.4) \cdot 10^{-11}, \quad (11)$$

where statistical and systematic errors are separately added in quadrature. In Table we present our results with statistical and systematic errors added in quadrature. Comparing Eq. (11) with the analogous calculation in [6] (see Table), one can see that our result is 23% more and the error is 2.5 times less. Note that increasing the result is caused mainly by ρ and ω interference in $\pi^0\gamma$ channel, which wasn't taken into account in previous works, where the Breit-Wigner formula for the cross

Contribution to $a_\mu \cdot 10^{11}$

State	Our value	Ref. [6]
$\pi^0\gamma$	46.0 ± 1.4	37 ± 3
$\eta\gamma$	7.1 ± 0.3	6.1 ± 1.4
$\pi^0\gamma + \eta\gamma$	53.1 ± 1.5	43 ± 4
hadrons+γ, total	95.7 ± 2.1	93 ± 11

section was used. The contribution (11) accounts for 133% of the projected error of the E821 experiment at Brookhaven National Laboratory ($40 \cdot 10^{-11}$) or 35% of the reached accuracy ($151 \cdot 10^{-11}$ [7]).

We can also take into account the intermediate state $\pi^0 e^+e^-$, using the obvious relation

$$\sigma(e^+e^- \to \pi^0 e^+ e^-, s) = \qquad (12)$$

$$= \frac{2}{\pi} \int\limits_{2m_e}^{\sqrt{s}-m_{\pi^0}} \frac{dm}{m^2} \Gamma_{\gamma* \to e^+e^-}(m) \sigma(e^+e^- \to \pi^0\gamma*, s, m),$$

where m is the invariant mass of the e^+e^- system,

$$\Gamma_{\gamma* \to e^+e^-}(m) = (1/2)\alpha\beta_e m(1-\beta_e^2/3),$$

$$\beta_e = \sqrt{1-4m_e^2/m^2},\ \sigma(e^+e^- \to \pi^0\gamma*, s, m) =$$

$$= \Big(p(m)/p(0)\Big)^3 \sigma(e^+e^- \to \pi^0\gamma, s),$$

$$p(m) = (\sqrt{s}/2)\sqrt{(1-(m_{\pi^0}+m)^2/s)(1-(m_{\pi^0}-m)^2/s)}$$

is the momentum of $\gamma*$ in s.c.m.

In the same way we can calculate $a_\mu(\mu^+\mu^-\pi^0)$ and $a_\mu(e^+e^-\eta)$. The result is

$$a_\mu(e^+e^-\pi^0) + a_\mu(\mu^+\mu^-\pi^0) + a_\mu(e^+e^-\eta) =$$

$$= (0.4 + 0.026 + 0.057) \cdot 10^{-11} = 0.5 \cdot 10^{-11}. \qquad (13)$$

Note that if $m \gtrsim m_\rho$ we have the effect of the excitation of resonances in the reaction $e^+e^- \to \pi^0(\rho,\ \omega) \to \pi^0 e^+e^-$. However this effect increases the final result (13) less than by 10% because of the factor $(p(m)/p(0))^3$, which suppresses the high m. So we ignore this correction. We also neglect $a_\mu(\mu^+\mu^-\eta) = 2 \cdot 10^{-14}$.

As it was noted in [6] and [8], it is necessary to take into account also

$$a_\mu(\text{hadrons} + \gamma, \text{rest}) = a_\mu(\pi^+\pi^-\gamma) +$$

$$+ a_\mu(\pi^0\pi^0\gamma) + a_\mu(\text{hadrons} + \gamma,\ s > 1.2\ \text{GeV}^2).$$

We take $a_\mu(\pi^+\pi^-\gamma) = (38.6 \pm 1.0) \cdot 10^{-11}$ from [8] (see also [6]), $a_\mu(\pi^0\pi^0\gamma) + a_\mu(\text{hadrons} + \gamma,\ s > 1.2\ \text{GeV}^2) = (4 \pm 1) \cdot 10^{-11}$ from [6]. Adding this to (11), we get

$$a_\mu(\text{hadrons} + \gamma,\ \text{total}) = (95.7 \pm 2.1) \cdot 10^{-11}. \qquad (14)$$

The contribution (14) accounts for 239% of the projected error of the E821 experiment or 63% of the reached accuracy.

In fact, the errors in (11) and (14) are negligible for any imaginable $(g-2)_\mu$ measurement in near future.

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