# Momentum space topology of fermion zero modes on brane

*G. E. Volovik*

*Low Temperature Laboratory, Helsinki University of Technology, FIN-02015 HUT, Finland*

*L.D. Landau Institute for Theoretical Physics RAS, 117940 Moscow, Russia*

Submitted 10 December 2001

We discuss fermion zero modes within the 3+1 brane – the domain wall between the two vacua in 4+1 spacetime. We do not assume relativistic invariance in 4+1 spacetime, or any special form of the 4+1 action. The only input is that the fermions in bulk are fully gapped and are described by nontrivial momentum-space topology. Then the 3+1 wall between such vacua contains chiral 3+1 fermions. The bosonic collective modes in the wall form the gauge and gravitational fields. In principle, this universality class of fermionic vacua can contain all the ingredients of the Standard Model and gravity.

PACS: 04.50.+h, 11.25.Mj, 11.27.+d, 73.43.–f

**Introduction.** The idea that our Universe lives on a brane embedded in higher dimensional space [1] is popular at the moment. It is the further development of an old ideas of extra compact dimensions introduced by Kaluza [2] and Klein [3]. In a new approach the compactification occurs because the low-energy physics is concentrated within the brane, for example, in a flat 4-dimensional brane embedded in a 5-dimensional anti-de Sitter space with a negative cosmological constant [4]. Branes can be represented by topological defects, such as domain walls (membranes) and strings. It is supposed that we live inside the core of such a defect. This new twist in the idea of extra dimensions is fashionable because by accomodation of the core size one can bring the gravitational Planck energy scale close to TeV range. That is why, there is a hope that the deviations from the Newton's law can become observable already at the distance of order 1 mm. At the moment the Newton's law has been tested for distances $> 0.2$ mm [5].

The particular mechanism of why the matter is localized on the brane, is that the low-energy fermionic matter is represented by the fermion zero modes, whose wave function is concentrated in the core region. Outside the core the fermions are massive and thus are frozen out at low temperature $T$. An example of such topologically induced Kaluza-Klein compactification of the multidimensional space is provided by the condensed matter analogs of branes – domain walls and vortices. These topological defects do contain fermion zero modes which can live only within the core of defects. These fermions form the 2+1 world within the domain wall, and 1+1 world within the domain wall in quasi-two-dimensional thin films or in the core of the linear defects – quantized vortices.

Recently an attempt was made to 'construct' the 4+1 condensed matter system with gapped fermions in bulk and gapless excitations on the 3+1 boundary, which include gauge bosons and gravitons [6]. This is the 4+1 dimensional generalization of the 2+1 quantum Hall effect (QHE) in the presence of the external $SU(2)$ gauge field, where the low energy fermions are the analogs of the so-called edge states [7] on the boundary of the system.

Here we show that there is a natural scenario in which the chiral fermions emergently appear in the brane together with gauge and gravitational field. Instead of the QHE system, we consider the system in which quantization of Hall conductivity occurs without external magnetic field. In this scenario the topology of momentum space [8, 9] plays the central role determining the universality classes. We consider the domain wall, which separates two 4+1 quantum vacua with nontrivial topology in the momentum space. If the momentum-space topological invariant is different on two sides of the wall, such 3+1 brane contains fermion zero modes – the gapless 3+1 fermions. Close to the nodes in the energy spectrum – Fermi points – these fermions are chiral. The collective bosonic modes within the brane correspond to gauge and gravitational fields acting on fermion modes.

As distinct from the relativistic theories [4], in this scenario the existence of gauge and gravitational fields in brane does not require the existence of the corresponding 4+1 fields in the bulk. The 3+1 fields in the brane emergently arise as collective modes of fermionic vacuum in the same manner as they arise in quantum liquids belonging to the universality class of Fermi points [8]. Thus, the brane separating the 4+1 vacua with different momentum-space topology is one more universality class of the quantum vacua, whose properties are dictated by the momentum-space topology.

**Walls in 2+1 systems.** Let us first consider how all this occurs in 2+1 systems, then it can be easily generalized to the 4+1 case. For the 2+1 systems it is known that the quantization of Hall or spin-Hall conductivity can occur even without an external magnetic field. This quantization is provided by the integer valued momentum-space topological invariant [10]:

$$N_3 = \qquad (1)$$
$$= \frac{1}{24\pi^2} e_{\mu\nu\lambda} \mathrm{tr} \int dp_x dp_y dp_0 \mathcal{G} \partial_{p_\mu} \mathcal{G}^{-1} \mathcal{G} \partial_{p_\nu} \mathcal{G}^{-1} \mathcal{G} \partial_{p_\lambda} \mathcal{G}^{-1}.$$

Here $\mathcal{G}$ is the fermionic propagator expressed in terms of the momentum $p_\mu = (\mathbf{p}, p_0)$, where $\mathbf{p} = (p_x, p_y)$ and $p_0$ is the frequency on the imaginary axis. In the most simple examples, which occur for example in thin films of $^3$He and probably in the atomic layers of some superconductors, one has $\mathcal{G}^{-1} = z - \mathcal{H}(\mathbf{p})$, where $z = ip_0$; and the $2 \times 2$ Hamiltonian $\mathcal{H}(\mathbf{p}) = \tau^i g_i(p_x, p_y)$ is expressed in terms of Pauli matrices $\tau^i$. In this case the Eq.(1) is simplified:

$$N_3 = \frac{1}{4\pi} \int \frac{dp_x dp_y}{|\mathbf{g}|^3} \mathbf{g} \cdot \left( \frac{\partial \mathbf{g}}{\partial p_x} \times \frac{\partial \mathbf{g}}{\partial p_y} \right). \qquad (2)$$

The invariant exists only if the fermions are gapped, i.e. their energy $E(p_x, p_y) = |\mathbf{g}| \neq 0$. The value of Hall or spin-Hall conductivity depends on this invariant and that is why the quantization of conductivities occurs without external field. The invariant $N_3$ can be varied by varying the thickness of the film, instead of varying the magnetic field in conventional QHE. Similar invariants have been used in conventional QHE too, see [11, 12].

Another important property of the conventional QHE, which is reproduced by the system under discussion, is the existence of the edge states on the boundary of the system, or on the boundary separating vacua with different values of quantized conductivity. Let us consider the domain wall (the 1+1 brane) separating vacua with different topological invariants on the left and on the right side of the wall: $N_3(right)$ and $N_3(left)$. If $N_3(right) \neq N_3(left)$ one finds that there are fermion zero modes. These are the gapless branches $E(p_{\|})$, where $p_{\|}$ is the linear momentum along the wall. These branches cross zero energy when $p_{\|}$ varies. Close to zero energy the spectrum of the $a$-th fermion zero mode is linear:

$$E_a(p_{\|}) = c_a(p_{\|} - p_a). \qquad (3)$$

These fermion zero modes correspond to the chiral (left-moving and right-moving) gapless edge states in QHE. There is an index theorem which determines the algebraic number $\nu$ of the fermion zero modes, i.e. the number of modes crossing zero with positive slope (right-moving) minus the number of modes with negative slope (left-moving):

$$\nu = \sum_a \mathrm{sign}\ c_a. \qquad (4)$$

According to this theorem, which is similar to the Atiyah-Singer index theorem [13] relating the number of fermion zero modes with the topological charge of the gauge field configuration, one has [14]

$$\nu = N_3(right) - N_3(left). \qquad (5)$$

The crossing point $p_a$ on each branch is nothing but the Fermi surface in 1D momentum space $p_{\|}$. In general the Fermi surfaces can be described by the topological invariant $N_1$ expressed in terms of Green's function [8]

$$N_1 = \mathrm{Tr} \oint_C \frac{dl}{2\pi i} \mathcal{G}(p_0, p_{\|}) \partial_l \mathcal{G}^{-1}(p_0, p_{\|}) = \nu. \qquad (6)$$

Here $\mathcal{G}$ is the propagator for the 1+1 fermion zero modes; and the contour $C$ embraces all the points ($p_0 = 0, p_{\|} = p_a$) where the Green's function is singular. In the simplest case the propagator for the 1+1 fermion zero modes has the form

$$\mathcal{G}^{-1} = ip_0 - E_a(p_{\|}), \qquad (7)$$

and the contour $C$ embraces the point ($p_0 = 0, p_{\|} = p_a$) in momentum space. The equation $N_1 = N_3(right) - N_3(left)$ illustrates the topology of the dimensional reduction in the momentum space: the momentum-space topological invariant $N_3$ of the bulk 2+1 system gives rise to the 1+1 fermion zero modes described by the momentum-space topological invariant $N_1$.

**Branes in 4+1 systems.** Now we can move from 2+1 to 4+1 dimension. Let us suppose that we have the quantum liquid in 4+1 spacetime, which contains the 3+1 domain wall separating two domains, each with fully gapped fermions. Then everything can be obtained from the case of the quantum liquid in 2+1 spacetime just by increasing the dimension.

According to analogy with 2+1 systems, the 4+1 gapped fermions must have nontrivial momentum-space topology. Such topology is described by the invariant $N_5$ instead of $N_3$:

$$N_5 = C_5 e_{\mu\nu\lambda\alpha\beta}\ \mathrm{tr} \int d^5p\ \mathcal{G} \partial_{p_\mu} \mathcal{G}^{-1} \mathcal{G} \partial_{p_\nu} \mathcal{G}^{-1} \mathcal{G} \partial_{p_\lambda} \times$$
$$\times \mathcal{G}^{-1} \mathcal{G} \partial_{p_\alpha} \mathcal{G}^{-1} \mathcal{G} \partial_{p_\beta} \mathcal{G}^{-1}. \qquad (8)$$

Here $p_\mu = (p_0, \mathbf{p})$, where $\mathbf{p} = (p_1, p_2, p_3, p_4)$ is the momentum in 4D space; $p_0$ is the energy considered at imaginary axis: $z = ip_0$; and $C_5$ is proper normalization. It is the difference $N_5(right) - N_5(left)$ of invariants on both sides of the domain wall (extension of

$N_3(right) - N_3(left)$), which must give rise to the 3+1 fermion zero modes within the brane.

The relativistic example of the propagator with nontrivial invariant $N_5$ is provided by $\mathcal{G}^{-1} = ip_0 - \mathcal{H}$, where the Hamiltonian in the 4D space is $\mathcal{H} = M\Gamma^5 + \sum_{i=1}^4 \Gamma^i p_i$, and $\Gamma^{1-5}$ are $4 \times 4$ Dirac matrices satisfying the Clifford algebra $\{\Gamma^a, \Gamma^b\} = 2\delta^{ab}$. In this example the proper domain wall, which contains the fermion zero modes, separates the domains with opposite signs of the mass parameter $M$, since for such wall $N_5(right) = -N_5(left)$. The existence of fermion zero modes in such domain wall is a well known fact in relativistic theories. We would like to stress, however, that the existence of fermion zero modes does not require the relativistic theory in the bulk. It is enough to have the nontrivial invariant $N_5$, which determines the universality class of fermionic vacuum.

In the 1+1 wall the energy spectrum of fermion zero modes in the wall crosses zero at points in 1D momentum space. Thus the energy spectrum of fermion zero modes in the 3+1 brain must be zero at points in 3D momentum space. This means that the spectrum has Fermi points. Fermi points are described by the momentum-space topological invariant $N_3$, which is now the difference between the number of right-handed and left-handed fermions [8]:

$$N_3 = \frac{1}{24\pi^2} e_{\mu\nu\lambda\gamma} \,\mathrm{tr} \int_{\sigma_3} dS^\gamma \; \mathcal{G}\partial_{p_\mu}\mathcal{G}^{-1}\mathcal{G}\partial_{p_\nu}\mathcal{G}^{-1}\mathcal{G}\partial_{p_\lambda}\mathcal{G}^{-1}. \tag{9}$$

Here the integral is over the 3-dimensional surface $\sigma_3$ embracing the singular points ($p_0 = 0, \mathbf{p} = \mathbf{p}_a$) of the spectrum. This is the analog of invariant $N_1$ in Eq.(6). Close to the $a$-th Fermi point the fermion zero modes represent 3+1 chiral fermions, whose propagator has the general form expressed in terms of the tetrad field:

$$\mathcal{G}^{-1} = \sigma^\nu e^\mu_{\nu a}(p_\mu - p_{\mu a}). \tag{10}$$

Here $\sigma^\nu = (1, \sigma)$, and $\sigma$ are Pauli matrices.

In the same manner as in Eqs.(5) and (6) which relate the number of fermion zero modes with the topological invariants in bulk 2+1 domains, the total topological charge of the Fermi points within the domain wall is expressed through the difference of the topological invariants in bulk 4+1 domains:

$$N_3 = N_5(right) - N_5(left)\ . \tag{11}$$

The quantites $e^\mu_{\nu a}$ and $p_{\mu a}$, which enter the fermionic spectrum, are dynamical variables. These are the low energy collective bosonic modes which play the part of the effective gravitational and gauge fields correspondingly acting on chiral fermion [8]. These fields emergently arise in the fermionic vacuum with nontrivial momentum-space topology. The brane between the topologically different vacua thus represents one more universality class of the 'emergent behavior' [15].

In a similar manner the gauge and gravity fields arise as collective modes on the boundary of the 4+1 system exhibiting the quantum Hall effect [6]. Both systems have similar topology: in Ref.[6] the nontrivial topology is provided by the external field, while in our case it is assumed that the vacuum itself has a nontrivial topology, $N_5 \neq 0$, even without the gauge field.

**Conclusion.** We showed that if the momentum-space topology of the fermionic vacuum in 4+1 space-time is nontrivial, the 3+1 domain wall between the two such vacua contains chiral fermions, while bosonic collective modes in the wall are the gauge and gravitational fields. This emergent behavior does not depend on details of the action in the bulk 4+1 systems, or on details of the structure of the brane. Neither Lorentz invariance, nor the gravity in 4+1 bulk system are required for emergency of chiral fermions and collective fields in the brane. However, the nontrivial topology alone does not guarantee that the gravitational field will obey Einstein equations: the proper (maybe discrete) symmetry and the proper relations between different "Planck" scales in the underlying fermionic system are required [8]. The energy scale which marks the cut-off of the integrals over fermions must be much smaller than the energy scale at which the Lorentz invariance is violated. We hope that within this universality class one can obtain all the ingredients of the Standard Model and gravity.

I thank V.A. Rubakov and S. Zhang for fruitful discussions. This work was supported by ESF COSLAB Programme and by the Russian Foundations for Fundamental Research.

1. V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **B125**, 136 (1983); K. Akama, in: Lecture Notes in Physics. *Gauge Theory and Gravitation* **176**, Eds. K. Kikkawa, N. Nakanishi, and H. Nariai, Springer-Verlag 1983, pp. 267–271 (K. Akama, An early proposal of 'brane world', hep-th/0001113).
2. T. Kaluza, Preus. Akad. Wiss. **K1**, 966 (1921).
3. O. Klein, Z. Phys. **37**, 895 (1926).
4. L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
5. C. D. Hoyle, U. Schmidt, B. R. Heckel et al., Phys. Rev. Lett. **86**, 1418 (2001).
6. S. C. Zhang and J. Hu, Sciense **294**, 823 (2001).

7. B. I. Halperin, Phys. Rev. **B25**, 2185 (1982); X. G. Wen, Phys. Rev. Lett. **64**, 2206 (1990); M. Stone, Phys. Rev. **B42**, 8399 (1990).
8. G. E. Volovik, Phys. Rep. **351**, 195 (2001).
9. G. E. Volovik, in: *Proc. of XVIII CFIF Autumn School "Topology of strongly correlated systems*, Lisbon 8–13 October 2000, World Scientific, Singapore, 2001, p. 30; cond-mat/0012195.
10. G. E. Volovik, and V. M. Yakovenko, J. Phys.: Cond. Matter **1**, 5263 (1989).
11. M. Kohmoto, Ann. Phys. **160**, 343 (1985).
12. K. Ishikawa and T. Matsuyama, Z. Phys. **C33**, 41 (1986); Nucl. Phys. **B280**, 523 (1987); T. Matsuyama, Progr. Theor. Phys. **77**, 711 (1987).
13. M. F. Atiyah and I. M. Singer, Annals. Math. **87**, 484 (1968); **93**, 119 (1971).
14. G. E. Volovik, JETP Lett. **66**, 522 (1997).
15. R. Laughlin and D. Pines, Proc. Natl. Acad. Sc. USA **97**, 28 (2000).