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On the Aizenman exponent in critical percolation

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The probabilities that clusters span a hypercube of dimensions two to seven along one axis of a percolation system under criticality were investigated numerically. We used a modified Hoshen–Kopelman algorithm combined with Grassberger's "go with the winner" strategy for the site percolation. We performed a finite-size analysis of the data and found that the probabilities confirm Aizenman's proposal for the multiplicity exponent for dimensions three to five. A crossover to the mean-field behavior around the upper critical dimension is also discussed.

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Percolation occurs in many natural processes from electrical conduction in disordered matter to oil extraction from a field. In the latter, the coefficient of oil extraction from the oil sands (the ratio of the actual extracted to the estimated oil) could be as much as 0.7 for light oil and as low as 0.05 for viscous heavy oil. An increase in this coefficient by any new point requires appreciable investment. Additional knowledge about the percolation model could reduce the amount of additional investment.

A remarkable breakthrough in the theory of critical percolation was established in the last decade thanks to a combination of mathematical proofs, exact solutions, and large-scale numerical simulations. A few years ago, Aizenman proposed a new exponent describing the probability $P(k,r)$ that the critical percolation d-dimensional system with the aspect ratio r is spanned by at least k clusters [1],

$$\ln P(k,r) \propto -\alpha_d\, k^{\zeta}\, r, \qquad (1)$$

where α_d is a universal coefficient that depends on only the universality class and $\zeta = d/(d-1)$.

In two dimensions, Aizenman's proposal (1) was proved mathematically [1], confirmed numerically [2], and derived exactly [3] using conformal field theory and Coulomb gas arguments. This exponent seems related to the exponents of two-dimensional copolymers [4]. In three dimensions, proposal (1) was checked numerically in [5] and more recently and more precisely in [6].

The upper critical dimension of percolation is $d_c = 6$, which follows from comparing the exponents derived on the Cayley tree with those satisfying scaling laws (see, e.g., [7] and [8]). The fractal dimension D_f of percolating critical clusters is equal to 4 above d_c, and the number of percolating clusters becomes infinite for $d > d_c$. This fact would imply that $\zeta = 0$ at $d = 6$ if we supposed (rather naively) that Aizenman's formula is applicable at the upper critical dimension. Supposing this is true and taking into account that the values of ζ for $d = 2$ and $d = 3$ are respectively 2 and 1.5, we could place all three points on the straight line $\zeta = (6-d)/2$ as depicted in Fig.1. We could then estimate the respective values of ζ

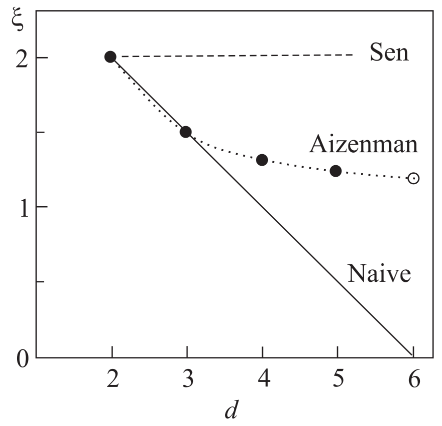

Fig.1. Variation of Aizenman exponent ζ with the space dimension d as predicted by Aizenman (circles and dotted line), claimed by Sen (dashed line) and discussed in the text (solid line)

for $d = 4$ and $d = 5$ to be $\zeta = 1$ and $\zeta = 0.5$, and these values are far from those predicted by Aizenman's formula, which are 4/3 and 5/4 respectively. In contrast, based on simulations, Sen [9] claims that $\zeta = 2$ for all dimensions from two to five.

The main purpose of our simulations is estimate the exponents for dimensions from two to six with sufficient accuracy that for $d = 4$ and $d = 5$, we can distinguish between the values predicted by Aizenman's formula and

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by a naive application of cluster fractal-dimension arguments and a straight-line fit as discussed above.

In the rest of the paper, we briefly summarize the highlights of our study, then present some details of our research, and finally discuss the results for the Aizenman exponent and the physics of a crossover from the Aizenman picture to the mean-field picture.

Our main results can be summarized as follows:

1. *Modified combination of the Hoshen–Kopelman algorithm and Grassberger's strategy.* We use the Hoshen–Kopelman (HK) algorithm [10] to generate clusters and Grassberger's "go with the winner" strategy [6] to track spanning clusters. We add a new tag array in the HK algorithm that allows reducing the tag memory order from L^d to L^{d-1}, where L is the linear size of the hypercubic lattice. As a result, the amount of memory is about two orders less for large values of L, and the program is about four times faster – the complexity of the algorithm is compensated by less memory being needed for for swapping to and from the auxiliary array.

2. *Efficient realization of combined shift-register random number generators.* We use an exclusive-or ($\oplus$) combination z_n of two shift registers:

$$x_n = x_{n-9689} \oplus x_{n-5502},$$
$$y_n = y_{n-4423} \oplus y_{n-2325}, \qquad z_n = x_n \oplus y_n \tag{2}$$

(see [11] and the references therein for a discussion). We reduce computational time for generating random numbers by the factor 3.5 by an efficient technical modification: we use the SSE command set that is available on processors of the Intel and AMD series starting from the Intel Pentium III and AMD Athlon XP.

3. *Extraction of the exponents for dimensions three to five.* We first use finite-size analysis to estimate the logarithm of the probability $P(k,r)$ in the limit of an infinite lattice size L. We then fit data as a function of the number of spanning clusters k to obtain the Aizenman exponent ζ.

4. *Confirmation of Aizenman's proposal.* The estimates of the exponent ζ for the dimensions $d = 2, 3, 4$, and 5 coincide well with those proposed by Aizenman.

5. *Qualitative interpretation of Aizenman's conjecture.* Cardy interpreted Aizenman's result qualitatively in two dimensions based on the assumption that the main mechanism for reducing the number of percolation clusters is some of them terminating. The same result could be derived for the mechanism of cluster confluence (or merging). This means that in low dimensions, the percolation clusters consist of a number of closed paths (loops), while in higher dimensions, clusters are more similar to trees. Indeed, it is well known that the probability of obtaining a loop becomes smaller for higher dimensions and goes to zero in the limit of infinite dimensions (Cayley tree) [7, 8].

6. *Crossover to mean-field behavior.* We found evidence that the probability that clusters span a hypercubic lattice goes to unity in the limit of high dimensions as follows from the well-accepted picture. We did not find any sharp changes in the probabilities around the upper critical dimension $d_c = 6$ but rather evidence for a crossover. Therefore, Aizenman's formula (1) could also be applied in dimensions higher (but not too much higher) than the upper critical dimension to describe the probabilities of spanning clusters in large, but finite, system sizes approximately.

We follow with the details of the critical percolation, simulations, and data analysis.

Spanning probability. We can define the probability $P(k,r;L)$ that k clusters traverse a d-dimensional hyperrectangle $[0,L]^{d-1} \times [0,Lr]$ in the Lr direction [1]. Provided that the scaling limit exists (this was proved recently by Smirnov for percolation in the plane [12]), the probability $P(k,r)$ could be defined as the limit of $P(k,r;L)$ as $L \to \infty$. Aizenman proposed that $P(k,r)$ should behave according to (1) in dimensions from three to five. The validity of formula (1) was well established [1, 3, 2].

Numerical results ([5] and [6]) for the exponent ζ for critical percolation on cubic lattices seems to confirm Aizenman's proposal for the value of $\zeta = 1.5$

Actually, we could consider the probability $P(k,r)$ as the probability of obtaining k clusters at the distance r from the left side of the hyperrectangle when clusters grow to the right. Only two processes could change the number of clusters, cluster merging and cluster terminating.

The differential dP of the probability would be

$$dP \propto P(k,r)\, k^{1/(d-1)}\, k\, dr, \tag{3}$$

where the right-hand side represents the product of the probability $P(k,r)$ and the differential of the total border hyperarea of k clusters, each with the hyperarea differential $k^{1/(d-1)}\, dr$. This expression follows from the area unit of measure being proportional to the characteristic transversal length of "infinite" clusters. Therefore, the transversal area remains constant as k changes, while the longitudinal length increment in these units is $\propto k^{1/(d-1)} dr$. Integrating (3), we recover probability (1). Thus, $P(k,r)$ describes the probability that k clusters would not merge together.

The same probability could be obtained by the process of cluster terminating as given by Cardy in the plane [3], which can be easily extended to dimensions $d > 2$.

This means that the exponent ζ cannot be larger than that proposed by Aizenman, and $\zeta = d/(d-1)$ is the upper bound for the exponent.

Algorithms and realizations. The classical realization of the HK algorithm [10] requires memory for two major structures, an array to keep a $(d-1)$-dimensional cluster slice and a tag array. The total memory required by the algorithm is $\propto L^{d-1} + p_c r L^d$, where p_c is the site percolation threshold value. Therefore, for large rL, a main advantage of the HK algorithm (i.e., relatively low memory consumption) is negated by the second term. Our modification of the original algorithm allows reducing memory for the tag array to about $3p_c L^{d-1}$.

Instead of keeping all the tags in memory and selecting new tags with increasing tag numbers, we create two arrays, of which one keeps the tag value and the other one keeps the number N of the slice where the corresponding tag was last used. When we build a cluster, we update this array with $N = N_{\rm current}$ for the tags used. If $N < N_{\rm current} - 1$, then this tag is not on the front surface of the sample and will never be used again, and we can therefore reuse it. We note that cluster size information should be taken into account before reusing the associated tag if a size information is required.

We use the "go with the winner" strategy [6] as follows. If the system has k spanning clusters for some aspect ratio $r = n\delta r$, it is stored in memory and grown for δr. If the resulting configuration then has k spanning clusters, it is stored, and the growth process continues. Otherwise, we return to the previously saved state. Using this procedure, we calculate the probability $P_i(\delta r)$ that the system propagates at the distance δr from the position $r = (i-1)\delta r$. Finally, we obtain $P(r = n\delta r) = \prod_{i=1}^{n} P_i(\delta r)$. Choosing sufficiently small values of δr, we can achieve rather high probabilities of $P_i(\delta r)$ (which can be determined from a few realizations), while the total probability might be very small (down to $\propto 10^{-100}$ in our case).

The random number generator was optimized for the SSE instruction set as follows. Because the length of all four RNG legs is $\{a|b\}_{\{x|y\}} > 4$, the nth step of the RNG does not intersect with the $(n+3)$th step. Therefore, we can pack four consecutive 32-bit values of $\{x_{n-\{a_x|b_x\}}\}$ and $\{y_{n-\{a_y|b_y\}}\}$ into 128-bit XMM registers, process them simultaneously (see Eq. (2)), and thus obtain z_n, z_{n+1}, z_{n+2}, and z_{n+3} within one RNG cycle.

Data analysis. The lattice size was varied from $L_{\rm min}$ to $L_{\rm max}$ with the step δL. In Table 1, particular values of the simulation parameters are presented together with the interval of the number of clusters k depending on the dimension d. The direct result of the simulations is the probabilities $P(k, r; L)$ that exactly k clusters connect two opposite surfaces (separated by the distance rL) of the rectangle with the size L^{d-1} in the "perpendicular" direction in which we apply periodic boundary conditions. We use values of the site percolation thresholds on the hypercubic lattices from [13]–[16] as shown in Table 1.

Таблица 1

d	k	$L_{\rm min}$	$L_{\rm max}$	δL	p_c	Ref.
2	1-5	16	256	16-32	0.59274621(13)	[13]
3	1-6	8	64	4,8	0.3116080(4)	[14]
4	1-6	8	48-56	4,8	0.196889(3)	[15]
5	1-6	4	32,24	4,8	0.14081(1)	[15]
6	1-6	4	15-16	3-5	0.109017(2)	[16]
7	1-4	4	10	1	0.0889511(9)	[16]

Minimal $L_{\rm min}$ and maximal $L_{\rm max}$ linear sizes of the percolation lattice and the interval δL between two consecutive values of L depending on the dimension d and number of clusters k. The values of p_c are taken from the references in the last column

Data analysis consists of three steps. First, we compute the slope $s(L)$ of $\ln P(k, r; L)$ for a given dimension d, number of clusters k, and linear lattice size L. An example of such a function is given in Fig.2 for $\ln P(5, r; 16)$ in the dimension four. We also plot the logarithm of the probability $P_+(k, r; L) = \sum_{k' \geqslant k} P(k', r; L)$ of the event that *at least* k clusters span the (hyper)rectangle at the distance rL. To calculate $s(L)$, we use data only in the interval of the aspect ratio r between 1.5 and 5. We note that the probability that five clusters span a rectangle with the linear size

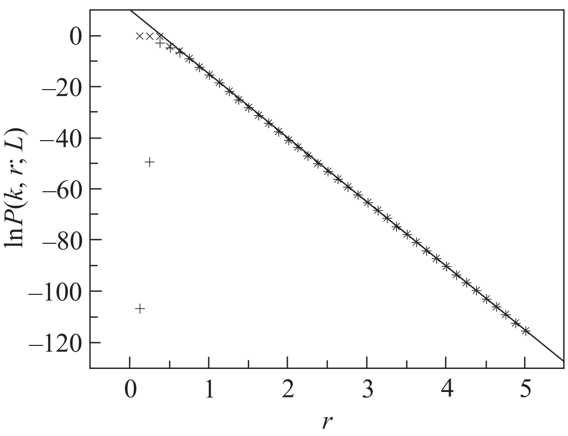

Fig.2. The logarithms of the probabilities of exactly k clusters $P(k, r; L)$ (+) and of at least k clusters $P_+(k, r; L)$ (×) for the dimension $d = 4$ and the number of clusters $k = 5$ as functions of the aspect ratio r. The linear size of the hyperrectangle is $L = 16$. The solid line is the linear approximation to $\ln P(k, r; L)$ on the interval $r = [1.5; 5.0]$

$L = 16$ at the distance $5 \cdot 16 = 80$ is extremely small $\approx 10^{-52}$.

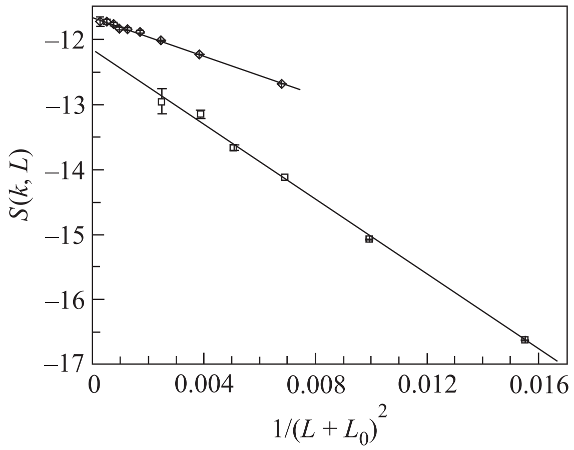

Fig.3. Plot of $s(k;l)$ for $k=3$ clusters in the dimension four as a function of $1/(L+L_0)$ with the fitting parameter $L_0 = 4.12$ ($\square$) and for $k = 4$ clusters in the dimension six with $L_0 = 4.03$ ($\diamond$). Straight lines result from the fitting to the corresponding data as discussed in the text

Second, we compute probabilities in the limit of an infinite system size L, fitting slopes $s(k)$ with the expression (see Fig.3)

$$s(k;L) = s(k) + \frac{B}{(L+L_0)^t}, \tag{4}$$

where B, t, and L_0 are fitting parameters [17, 5, 18]. The resulting values of the slopes $s(k)$ are presented in Table 2. The number of runs used to compute each particular entry in the Table 2 varied from 10^6 to several tens for higher dimensions.

Таблица 2

	d				
k	3	4	5	6	7
1	-1.377(1)	-1.774(3)	-1.859(9)	-1.76(2)	-1.48(4)
2	-6.919(6)	-6.330(15)	-5.57(6)	-4.73(8)	-3.55(11)
3	-13.655(15)	-11.64(4)	-9.95(12)	-8.27(12)	-6.25(16)
4	-21.47(3)	-17.77(6)	-14.65(20)	-11.95(25)	-9.3(3)
5	-30.23(3)	-24.02(8)	-19.9(3)	-15.75(30)	
6	-40.02(6)	-31.0(1)	-25.0(3)	-22.7(2)	

Values of $s(k)$ for different numbers of clusters k and dimensions d for site percolation on hypercubic lattices with periodic boundary conditions in directions perpendicular to the spanning direction

We checked the accuracy of our simulations as well as the validity of the approach in general for site percolation on a square lattice. Table 3 shows a comparison of our results for the slope s with the exact values and with early simulations, in which another modification of the HK algorithm, but not the Grassberger strategy, was used. We note that our results coincide well with the exact ones and give a higher accuracy for larger values of k in comparison with the previous numerical results despite the smaller computation time used. Our data for $k = 1$ is less accurate because of the smaller statistics we used (10^6 runs as compared with 10^8 samples in [5]). This is a direct demonstration of the effectiveness of the Grassberger strategy for large values of k.

Таблица 3

k	this paper	exact from [3]	from [5]
1	-0.6541(5)	-0.6544985	-0.65448(5)
2	-7.855(3)	-7.85390	-7.852(1)
3	-18.32(1)	-18.3260	-18.11(15)
4	-32.99(3)	-32.9867	
5	-51.83(2)	-51.8363	

Values of $s(k)$ in two dimensions for different k calculated in this paper, using the exact Cardy formula [3], and estimated in [5] for site percolation on a tube

Finally, we use values in Table 2 to determine the Aizenman exponent ζ, fitting data in each column with

$$s = A(k^2 - k_0)^{p/2} \tag{5}$$

in two and three dimensions as proposed by Grassberger [6] and with

$$s = A(k^p - k_0) \tag{6}$$

in higher dimensions. Here A, k_0, and p are fitting parameters. We take only the leading behavior in k into account.

Таблица 4

d	A	k_0	p
2	2.090(4)	0.244(5)	2.0012(10)
	2.0940(5)	0.2489(7)	2
3	2.81(4)	0.64(4)	1.489(7)
	2.757(2)	0.587(3)	3/2
4	3.06(20)	0.41(6)	1.315(30)
	2.949(5)	0.373(3)	4/3
5	2.8(1)	0.40(4)	1.24(3)
	2.78(2)	0.38(2)	5/4
6	2.8(8)	0.5(3)	1.12(14)
	2.41(5)	0.33(6)	6/5
7	1.4(10)	0.08(116)	1.4(4)
	2.03(12)	0.50(13)	7/6

Values of the fitting parameters A and k_0 and the power p as defined in Eqs. (5) and (6) for the dimension d

Spanning, proliferation, and crossover to mean-field behavior. The results of the final fit with (5) and (6) are shown in Table 4. The second row for each particular dimension d is the fit with the value of power p fixed to the Aizenman exponent value. This is done to check the fit stability. Indeed, the values of A and k_0 coincide within one standard deviation for the dimensions two to five.

The larger deviations of parameters for the dimensions six and seven can be attributed to cluster proliferation appearing–the number of clusters is known [1] to grow as L^{d-6} in dimensions $d > d_c = 6$. We plot the coefficient α_d (defined by expression (1)) in Fig.4 as a function of the dimension d. The probability for exactly one cluster to span at given distance r becomes smaller as the dimension increases from two to five and larger for larger dimensions as can be seen from the first row ($k = 1$) of Table 2 and from the lower curve dependence in Fig.4. For any fixed d, the value of α_d approaches some limit for the dimensions two to five and $k > 2$, which suggests the value of the corrections to the leading behavior in k (see Eqs. (5) and (6)).

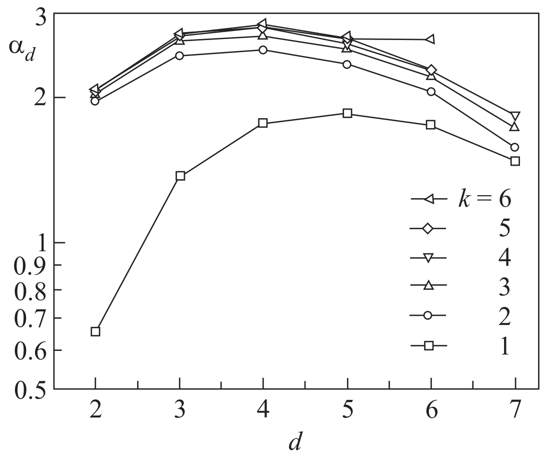

Fig.4. The coefficient α_d (as a function of the dimension d) extracted from the probabilities $P(k, r)$ for different numbers of clusters k

That the value of ζ we formally extracted from our data for $d = 6$ more or less coincides with $\zeta = d/(d-1) = 6/5$ as formally computed from the Aizenman expression could be interpreted to mean that the number of clusters depends logarithmically on the lattice size L. One could expect the logarithmic behavior to be visible only for somewhat larger values of L than we have so far used (see Table 1). With the values of L of the order we have used in simulations, we see effectively the same picture as for the lower dimensions–clusters spanning according to the Aizenman formula. This means that at small (or moderate) values of L, the main mechanism is as discussed above, cluster merging and terminating. And only at sufficiently large system sizes will we see cluster proliferation. An indication of that can be seen from the values of α_d in the dimension seven in Fig.4. The probabilities become closer, and this could be attributed to cluster proliferation and treated as a crossover to the mean-field behavior.

Discussion. The results show the validity of Aizenman's proposal in the dimensions three to five (results in the plane were already proved rigorously) and did not support Parongama Sen claims based on heir simulations (Fig.1). We found evidence for cluster proliferation for the dimension seven. The analysis could be extended for the number of spanning clusters to distinguish exponential decay with the system size of the number of clusters for the dimension five, logarithmic growth of them for the dimension six, and linear growth for the dimension seven. The same technique could be used to establish such a crossover to the mean-field picture numerically although significantly more computational time than we have used would be needed for this. In fact, the linear growth of multiplicity of the spanning clusters at seven dimensional critical percolation was confirmed numerically in the preprint [19] posted at *arXiv* preprint library a few days after the present one, cond-mat/0207605.

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