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Anticommutativity equation in topological quantum mechanics

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We consider the topological quantum mechanics as an example of topological field theory and show that it special properties lead to numerous interesting relations for topological corellators in this theory. We prove that the generating function $\mathcal{F}$ for this correlators satisfies the anticommutativity equation $(\mathcal{D}-\mathcal{F})^2=0$. We show that commutativity equation $[dB,dB]=0$ could be considered as a special case of anticommutativity equation.

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1. During last two decades there have been much interest to quantum field theories whose corellators do not depend on points and metric, but only on topology of the manifold and on the types of operators in the corellators. These theories are called topological[1]. Among the most celebrated examples are the Chern-Simons theory [2], $N=2$ twisted gauge theories [3], topological sigma models [4]. Here we consider yet another, much simpler example of topological theory which is a subsector of the supersymmetric quantum mechanics. We called it topological quantum mechanics.

An explicit construction of many topological theories is given by a BRST-like symmetry operator Q, $Q^2=0$ so that the energy momentum tensor of the theory has the special form [3]

$$T_{\mu\nu}=\{Q,G_{\mu\nu}\}, \tag{1}$$

where $G_{\mu\nu}$ is some tensor. This formula leads to many interesting corollaries. In particular, it provides corellators with the independent of metric and points, which are topological invariants.

In the case of one dimensional theory, the energy momentum tensor has only one component and is equal to the Hamiltonian

$$H=T_{00}=\{Q,G_{00}\}=\{Q,G_+\}. \tag{2}$$

In topological quantum mechanics, the objects to study are corellators $<\Phi_{A_1}(t_1)...\Phi_{A_n}(t_n)>$ of Q-closed operators. They would naively depend of n times $t_1<...<t_n$. However, since the energy momentum tensor is anticommutator of Q and G_+, the corellator actually does not depend on t's and is given by a factorization formula

$$\begin{aligned}&<\Phi_{A_1}(t_1)...\Phi_{A_n}(t_n)>=\\&=<0|e^{-t_1H}\Phi_{A_1}e^{(t_1-t_2)H}\times\\&\times\Phi_{A_2}...e^{(t_{n-1}-t_n)H}\Phi_{A_n}e^{t_nH}|0>=\\&=\lim_{(t_k-t_{k-1})\to\infty}<0|\Phi_{A_1}e^{(t_1-t_2)H}...e^{(t_{n-1}-t_n)H}\times\\&\times\Phi_{A_n}|0>=<\Phi_{A_1}>...<\Phi_{A_n}>.\end{aligned} \tag{3}$$

It means that it is sufficient to study the corellators of one operator Φ_A in the theory with the deformed supercharge Q:

$$Q\to Q+\sum\Phi_AT_A=Q+\Phi. \tag{4}$$

Later we will see that this one-point corellator is a total derivative of $\mathcal{F}$ on the space of coupling constants T_A:

$$\partial_A\mathcal{F}=<\Phi_A>_{\text{deformed}}=<T\left\{\Phi_Ae^{\int\{\Phi,G_+\}dt}\right\}>, \tag{5}$$

where $T\{...\}$ stands for the chronological ordering. In what follows, we carefully formulate the theory and realize that the function $\mathcal{F}$ satisfies an interesting quadratic differential equation which we call anticommutativity equation.

2. Thus, we consider the simplest example of topological field theory, topological quantum mechanics. As usual, there is a nilpotent symmetry operator Q and one can introduce the Hamiltonian in the form (2). All the operators act on the space V with the following properties

$$\begin{gathered}V=V_1\oplus V_0,\\HV_0=0,\quad QV_0=0,\\H>0\ \text{ on }\ V_1.\end{gathered} \tag{6}$$

This physically means that V is a space of states of our system, V_0 is a space of the vacuum states. We assume that the kernel V_0 of the Hamiltonian consists of

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Q-closed states, and all non-zero energies have strictly positive real part. In this theory we study the corellators of the form (5), which are the vacuum matrix elements, i.e. the operators from $\mathrm{Hom}(V_0, V_0)$ and can be written in the following form[2)]

$$F^{(k)}_{\alpha\beta} = \int\limits_0^\infty ... \int\limits_0^\infty < 0_\alpha|\Phi G_+ e^{-t_1 H}\Phi G_+ e^{-t_2 H} ... \Phi|0_\beta > \times$$
$$\times\, dt_1..dt_{k-1} =$$
$$= \frac{1}{k!} < T\left\{\Phi(0)(\int\limits_{-\infty}^{+\infty}\{G_+, \Phi(t)dt\})^{k-1}\right\} >_{\alpha\beta}. \quad (7)$$

We want to present this operator from $\mathrm{Hom}(V_0, V_0)$ as an operator from $\mathrm{Hom}(V, V)$. The way to do this is to insert Π_0's at the beginning and at the end. After this, we finally obtain the object we needed

$$\mathcal{F}^{(k)} =$$
$$= \int\limits_0^\infty ... \int\limits_0^\infty \Pi_0 \Phi G_+ e^{-t_1 H}\Phi G_+ e^{-t_2 H} .... \Phi \Pi_0 dt_1..dt_{k-1}. \quad (8)$$

Here Φ is the generating function of the following operators

$$\Phi = \Phi(0) = \sum \Phi_A T_A. \quad (9)$$

Our theory is Euclidean, which means that the evolution Φ in time can be described by the following formula

$$\Phi(t) = e^{-tH}\Phi(0)e^{tH}. \quad (10)$$

Our operators can be both even and odd, therefore, we need to introduce a superalgebra for describing their properties, our coupling constants T_A being graded too. Their algebra is

$$\{T_A, T_B\}_s = 0, \quad (11)$$

where $\{\dots\}$ stands for supercommutator.

In order to get some non-trivial interesting properties of the correlators, let us consider a special set of operators that satisfy the following algebra

$$\begin{gathered}\{\Phi_A, \Phi_B\}_s = C^K_{AB}\Phi_K, \\ \{Q, \Phi_A\}_s = 0.\end{gathered} \quad (12)$$

Π_0 being a projector onto the vacuum states. One can rewrite Π_0 in terms of H

$$\Pi_0 = \lim_{t\to\infty} e^{-tH}. \quad (13)$$

[2)]Here we used the properties $G_+^2 = 0$ and $G_+ V_0 = 0$ that allow one to interpret F as a corellator in topological theory, but they are not necessary for our main result.

It commutes with Q. Therefore,

$$\begin{gathered}Q\Pi_0 = \Pi_0 Q = 0, \\ < 0_\alpha|Q =< 0_\alpha|\Pi_0 Q = 0.\end{gathered} \quad (14)$$

3. Algebra of operators Φ **generates numerous commutation relations** for $\mathcal{F}^{(k)}$ which can be written in a compact form in terms of the generating function $\mathcal{F}$:

$$\mathcal{F} = \sum \mathcal{F}^{(k)} \quad (15)$$

as

$$\{\mathcal{F}, \mathcal{F}\} = T_A T_B C^K_{AB}\partial_K \mathcal{F}. \quad (16)$$

One can easily obtain all our corellators as derivatives of $\mathcal{F}$. We now prove this equation in the generic case.

Let us introduce an operator-valued differential form of indefinite rank on R

$$U = e^{-tH} + G_+ dt e^{-tH}. \quad (17)$$

We can check that

$$dU + \{Q, U\} = 0. \quad (18)$$

One can construct the differential form on R^{k-1} which takes value in $\mathrm{Hom}(V, V)$ and includes $k+1$ Φ's

$$\omega^{(k-1)} = \Pi_0 \Phi U_1 ... \Phi U_k \Phi \Pi_0. \quad (19)$$

Our form is d-closed because $Q\Pi_0 = \Pi_0 Q = 0$ and Φ anticommutes with Q

$$\begin{gathered}d\omega^{(k-1)} = -\Pi_0\Phi\{U_1, Q\}...\Phi U_k \Phi\Pi_0 - \\ - ... - \Pi_0 \Phi U_1 ... \Phi\{U_k, Q\}\Phi\Pi_0 = 0.\end{gathered} \quad (20)$$

One can consider the integral of the rank $k-1$ component of our form over the boundary of some surface and rewrite it as an integral of $d\omega^{(k-1)}$ over this surface by the Stocks theorem. The surface of integration is the boundary of the k-dimensional cube. The result of integration is as follows

$$\int\limits_0^\infty .. \int\limits_0^\infty \Pi_0\Phi(\Pi_0 - 1)\Phi G_+ e^{-t_1 H} ... G_+ e^{-t_k H} \times$$
$$\times\, \Phi\Pi_0 dt_1..dt_{k-1} + (\text{perm. of } \Pi_0 - 1) = 0. \quad (21)$$

The terms that include Π_0 come from the commutator $\{\mathcal{F}, \mathcal{F}\}$, while other terms come from $T_A T_B C^K_{AB}\partial_K\mathcal{F}$. For our superalgebra of operators Φ_A, one can write (super)Jacobi identities in the following form

$$C^K_{AB} C^E_{KD} = 0. \quad (22)$$

If we introduce the derivative $\mathcal{D} = \frac{1}{2}T_A T_B C^K_{AB}\partial_K$, our anticommutativity equation can be written as zero curvature equation

$$\{\mathcal{D}-\mathcal{F},\mathcal{D}-\mathcal{F}\} = 0. \tag{23}$$

4. Let us consider what this equation gives for the linear part of $\mathcal{F}$ in (15), F_A. Then, (23) leads to

$$\{F_A, F_B\} = C^K_{AB}F_K. \tag{24}$$

For odd operators Φ_A's, this equation is quite nontrivial. One can see this by considering the matrix example. In this case, our space of states V is the four-dimensional vector space. The operators Q and G_+ are 4×4 matrices

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H = \{Q, G_+\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{25}$$

V_0 is the subspace of V with first two zero coefficients. The operators Φ_A's which satisfy commutation relation (15) are as follows

$$\Phi_A = \begin{pmatrix} a & 0 & 0 & 0 \\ * & -a & * & * \\ * & 0 & \multicolumn{2}{c}{\multirow{2}{*}{F_A}} \\ * & 0 & & \end{pmatrix} \tag{26}$$

where stars are some numbers. When considering commutation relations on Φ_A, we are interested only in the right bottom block. It is easy to show that in this block there is anticommutator of the two right bottom blocks of Φ_A and Φ_B. It is what we get commuting matrix elements F_A. This block is the only one remaining after acting of the two projectors onto Φ_A. Thus, we observe that this is only these special commutation relations of matrices Φ_A that provide us with the similar algebra for F_A (i.e. for the right bottom blocks)

$$\{\Phi_A, \Phi_B\} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & \multicolumn{2}{c}{\multirow{2}{*}{$\{F_A, F_B\}$}} \\ * & * & & \end{pmatrix}. \tag{27}$$

In the previous formula, one could see that no other block contributes to the right bottom block. Let us see this in terms of operators F_A. Consider

$$\{F_A, F_B\} = \Pi_0\Phi_A\Pi_0\Phi_B\Pi_0 + \Pi_0\Phi_B\Pi_0\Phi_A\Pi_0. \tag{28}$$

Using formula (13) for the projector, one arrives at the following formula

$$\Pi_0 = \lim_{t\to\infty} e^{-tH} = \lim_{t\to\infty}\sum(-t)^k\frac{H^k}{k!} = 1 + \tag{29}$$
$$+\left(\sum QG_+(-t)^k\frac{H^{k-1}}{k!} + \sum(-t)^k G_+\frac{H^{k-1}}{k!}Q\right)\Bigg|_{t\to\infty}.$$

An important property of this representation is

$$\begin{gathered} \Pi_0\Phi_A QG_+(-t)^k\tfrac{H^{k-1}}{k!}\Phi_B\Pi_0 = 0, \\ \Pi_0\Phi_A G_+(-t)^k\tfrac{H^{k-1}}{k!}Q\Phi_B\Pi_0 = 0. \end{gathered} \tag{30}$$

Hence, the only term in (28) survived is the c-number term in (29) and

$$\{F_A, F_B\} = \Pi_0\Phi_A\Phi_B\Pi_0 + \Pi_0\Phi_B\Phi_A\Pi_0 = C^K_{AB}F_K. \tag{31}$$

5. Our anticommutativity equation contains the commutativity equation introduced in [5–7] as a special case for special operators. Recall the commutativity equation have the form:

$$[B_\mu, B_\nu] = 0 \quad , \quad B_\mu = \frac{\partial B}{\partial\tau_\mu}. \tag{32}$$

Namely, consider a new odd operator G_- that satisfy:

$$G_-^2 = 0, \quad \{G_-, G_+\} = 0 \quad \{G_-, Q\} = 0 \quad G_-V_0 = 0. \tag{33}$$

Consider even operators Φ_μ which satisfy relations (12) with $C = 0$ and

$$[[\Phi_\mu, G_-], \Phi_\nu] = 0. \tag{34}$$

This properties allows us to add the new odd operators $\Phi'_\mu = [\Phi_\mu, G_-]$ to our algebra because they satisfy (12) and write $\mathcal{F}$ in their terms. The main object in commutativity equations in terms $\mathcal{F}$ is

$$B_\mu = \frac{\partial\mathcal{F}}{\partial\theta_\mu}|_{\theta=0} \tag{35}$$

there we introduce two kinds of coupling constants T_A: even τ_μ and odd θ_μ. Really one can show that (33) leads to

$$\begin{gathered} \mathcal{F}|_{\theta=0} = 0, \quad \text{so} \\ \frac{\partial}{\partial\theta_\mu}\frac{\partial}{\partial\theta_\nu}\{\mathcal{F},\mathcal{F}\}|_{\theta=0} = [\partial_\mu\mathcal{F}|_{\theta=0}, \partial_\mu\mathcal{F}|_{\theta=0}] = 0. \end{gathered} \tag{36}$$

It could be the commutativity equation if we will show that $B_\mu = \partial B/\partial\tau_\mu$. It follows from the properties of G_-:

$$B_\mu = \frac{\partial B}{\partial \tau_\mu} \Longrightarrow \frac{\partial B_\nu}{\partial \tau_\mu} = \frac{\partial B_\mu}{\partial \tau_\nu} \sim \qquad (37)$$

$$\sim ...\Phi_\mu G_+ e^{-tH}[\Phi_\nu, G_-]... = ...[\Phi_\mu, G_-]G_+ e^{-tH}\Phi_\nu...$$

6. Consider the deformation of the solutions for anticommutativity equation by variation of operator G_+ in terms of $\mathcal{K} = G_+/H = \int\limits_0^\infty G_+ e^{-tH} dt$, so that:

$$\{\mathcal{K}, Q\} = 1 - \Pi_0. \qquad (38)$$

The variation of $\mathcal{K}$ satisfy:

$$\{\delta\mathcal{K}, Q\} = 0 \Longrightarrow \delta\mathcal{K} = [Q, Z]. \qquad (39)$$

The variation $\delta\mathcal{K}$ is exact, since Q doesn't have cohomologies in V_1. We can write the variation of solution to the anticommutativity equation and after some algebra we get:

$$\delta_{\mathcal{K}}\mathcal{F} = -\{\mathcal{D} - \mathcal{F}, \mathcal{F}_Z\}_s. \qquad (40)$$

In (40) $\mathcal{F}_Z$ is obtained from expression for $\mathcal{F}$ in terms of[3] $\mathcal{K}$ by replacing of an insertion of $\mathcal{K}$ by Z. The variation of $\mathcal{F}$ in form (40) is an analog of the gauge transformations in zero curvature equation (23) because they retain this equation:

$$\begin{aligned} &\{\mathcal{D} - \mathcal{F}, \mathcal{D} - \mathcal{F}\} = 0 \Longrightarrow \{\mathcal{D} - \mathcal{F}, \delta\mathcal{F}\} = 0, \\ &\{\mathcal{D} - \mathcal{F}, \delta_{\mathcal{K}}\mathcal{F}\} = \{\mathcal{D} - \mathcal{F}, \{\mathcal{D} - \mathcal{F}, \mathcal{F}_Z\}_s\} = 0. \end{aligned} \qquad (41)$$

The special property of commutativity equation is that B_μ is invariant under the variations of $\mathcal{K}$, it follows from (40) and (33). Really $\mathcal{F}|_{\theta=0} = 0$ and $\mathcal{F}_Z|_{\theta=0} = 0$, so the variation term linear in θ is:

$$\delta_{\mathcal{K}} B_\mu = \delta_{\mathcal{K}} \frac{\partial \mathcal{F}}{\partial \theta_\mu}|_{\theta=0} = 0. \qquad (42)$$

In case of anticommutativity equation we have in general only two terms invariant under the variations of $\mathcal{K}$: $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$.

Thus, we considered the topological quantum mechanics as an example of topological field theory and shown that it special properties lead to numerous interesting relations for topological corellators in this theory. We prove that the generating function $\mathcal{F}$ for this correlators satisfies the anticommutativity equation $(\mathcal{D} - \mathcal{F})^2 = 0$. We show that commutativity equation $[dB, dB] = 0$ could be considered as a special case of anticommutativity equation.

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[3] Schematically, $\mathcal{F} = \sum \Pi_0 \Phi\mathcal{K}\Phi\mathcal{K}...\Phi\mathcal{K}\Phi\Pi_0$.