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Remarks on A_2 Toda theory

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We study the Toda field theory with finite Lie algebras using an extension of the Goulian-Li technique. In this way, we show that, after integrating over the zero mode in the correlation functions of the exponential fields, the resulting correlation function resembles that of a free theory. Furthermore, it is shown that for some ratios of the charges of the exponential fields the four-point correlation functions which contain a degenerate field satisfy the Riemann ordinary differential equation. Using this fact and the crossing symmetry, we derive a set of functional equations for the structure constants of the A_2 Toda field theory.

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1. The Toda field theory (TFT) provides an extremely useful description of a large class of two-dimensional integrable quantum field theories. For this reason these models have attracted a considerable interest in recent years and many outstanding results in various directions have been established.

TFTs are divided in three broad categories: finite Toda theories (FTFTs) for which the underlying Kac-Moody algebra [1, 2] is a finite Lie algebra, affine Toda theories (ATFTs) for which the underlying Kac-Moody algebra is an affine algebra and indefinite Toda theories (ITFTs) for which the underlying Kac-Moody algebra is an indefinite Kac-Moody algebra. The classes of FTFTs and ATFTs are well-studied and known to be integrable. In addition, the FTFTs enjoy conformal invariance. A review of the most interesting developments in ATFTs is presented in Ref. [3] where there is also a list of references to the original papers. The class of ITFTs is the least studied as there are still many open questions regarding the indefinite Kac-Moody algebras. A special class of the ITFTs, namely the hyperbolic Toda Theories (HTFTs), for which the underlying Kac-Moody algebra is a hyperbolic Kac-Moody algebra were studied in Ref. [4] and it was shown that they are conformal but not integrable.

However, despite all progress in TFTs, there still remain many unresolved questions and problems. For example, one may ask what the structure constants of the conformally invariant TFTs are. In this paper, we address this question. We focus on FTFs and, in particular, on the A_2 FTT.

In Sec. 2 the A_2 FTFT is introduced, some notations are fixed, and then we continue to show how the correlation function of exponential fields in the FTFT reduces to correlation functions of a free field theory with conformal W-symmetry [5 – 8]. In Sec. 3 we prove that, for some special cases of the exponential fields, the four-point correlation functions which contain a "degenerate" primary field satisfy the Riemann ordinary differential equation. Then, in Sec. 4 the conformal bootstrap technique is applied to derive a set of functional equations for the structure constants of the A_2 FTFT.

2. A_2 Finite Toda Field Theory. We consider the finite conformal Toda field theory associated with the simply-laced Lie algebra A_2 described by the action

$$S = \int d^2x \left[\frac{1}{8\pi}(\partial\boldsymbol{\varphi})^2 + \mu \sum_{i=1}^{2} e^{b\mathbf{e}_i\cdot\boldsymbol{\varphi}} + \frac{R}{4\pi}\mathbf{Q}\cdot\boldsymbol{\varphi} \right]. \quad (1)$$

In the above equation, $\mathbf{e}_i$ $(i = 1, 2)$ are the simple roots of Lie algebra A_2. These define the fundamental weights $\mathbf{w}_i$ of the Lie algebra by the equation

$$\mathbf{e}_i \cdot \mathbf{w}_j = \delta_{ij}.$$

The background charge $\mathbf{Q}$ is proportional to the Weyl vector $\boldsymbol{\rho}$:

$$\mathbf{Q} = (b + 1/b)\,\boldsymbol{\rho}, \qquad \boldsymbol{\rho} = \sum_{i=1}^{2} \mathbf{w}_i.$$

The local conformal invariance of the FTFT with central charge

$$c = 2 + 12\mathbf{Q}^2$$

is ensured by the existence of the holomorphic and antiholomorphic energy-momentum tensors

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$$T(z) = -\frac{1}{2}(\partial\varphi)^2 + \mathbf{Q}\cdot\partial^2\varphi,$$
$$\overline{T}(\bar z) = -\frac{1}{2}(\bar\partial\varphi)^2 + \mathbf{Q}\cdot\bar\partial^2\varphi.$$

It is well-known that the FTFTs possess, besides the standard conformal symmetry, an additional W-symmetry. In particular, the A_2 FTFT we are studying in the present paper contains the additional holomorphic and antiholomorphic currents $W(z)$, $\overline{W}(\bar z)$ with spin 3, which generate the W_3 algebra.

The vertex operators

$$V_{\mathbf{a}}(x) = e^{2\mathbf{a}\cdot\varphi(x)}$$

are spinless primary fields of the W-algebra. Let L_n, W_n be the Fourier modes of the holomorphic fields $T(z)$, $W(z)$. Then

$$L_0 V_{\mathbf{a}} = \Delta(\mathbf{a})\, V_{\mathbf{a}}, \quad W_0 V_{\mathbf{a}} = w(\mathbf{a})\, V_{\mathbf{a}},$$
$$L_n V_{\mathbf{a}} = 0, \qquad W_n V_{\mathbf{a}} = 0, \quad n > 0,$$

where the conformal dimension $\Delta(\mathbf{a})$ is given by

$$\Delta(\mathbf{a}) = 2\mathbf{a}\cdot(\mathbf{Q}-\mathbf{a}).$$

The correlation function of N vertex operators is formally defined by the functional integral

$$G_{\mathbf{a}_1,\dots,\mathbf{a}_n}(x_1,\dots,x_n) = \int \mathcal{D}\varphi \prod_{i=1}^{N} e^{2\mathbf{a}_i\cdot\varphi(x_i)} e^{-S[\varphi]}. \quad (2)$$

We introduce the following orthogonal decomposition of the field φ:

$$\varphi(x) = \varphi_0 + \tilde\varphi(x),$$

where φ_0 is the zero mode and $\tilde\varphi$ denotes the part of the field that is orthogonal to the zero mode:

$$\int d^2x\, \tilde\varphi(x) = 0.$$

Now, the integration of the functional integral (2) over the zero mode φ_0 can be done in a similar fashion to the Liouville case [9] to find

$$G_{\mathbf{a}_1,\dots,\mathbf{a}_n}(x_1,\dots,x_n) = = \left(\frac{\mu}{8\pi}\right)^{s_1+s_2} \frac{1}{b^2|\det e|}\Gamma(-s_1)\Gamma(-s_2)\times \times \int \mathcal{D}\tilde\varphi \prod_{i=1}^{N} e^{2\mathbf{a}_i\tilde\varphi(x_i)} \left(\int d^2x\, e^{b\mathbf{e}_1\cdot\tilde\varphi}\right)^{s_1} \times \times \left(\int d^2x\, e^{b\mathbf{e}_2\cdot\tilde\varphi}\right)^{s_2} e^{-S_0[\tilde\varphi]}, \quad (3)$$

where S_0 is the action of the free field theory,

$$S_0 = \int d^2x \left(\frac{1}{8\pi}(\partial\tilde\varphi)^2 + \frac{R}{4\pi}\mathbf{Q}\cdot\tilde\varphi\right),$$

and

$$s_1 = (b\det e_{ij})^{-1}[-Qe_{22} + k_1 e_{22} - k_2 e_{21}],$$
$$s_2 = (b\det e_{ij})^{-1}[-Qe_{12} + k_2 e_{11} - k_1 e_{12}],$$
$$\mathbf{k} = 2\sum_{i=1}^{N}\mathbf{a}_i, \qquad \mathbf{Q} = (Q,0).$$

Assuming that s_1 and s_2 are both positive integers, then the remaining functional integral in expression (3) can be reduced to the correlation function of the W_3 minimal model [7, 8]. Unfortunately, the situation is much more complicated, i.e., in general, s_1 and s_2 are not positive integers. However, the solution of the problem is hidden in the previous observation: supposing that we know the exact expressions of the structure constants for the W_3 minimal model, then we can recover the expressions for the structure constants of the A_2 FTFT by analytic continuation (similarly to the Liouville case) [10, 11].

3. Four-Point Correlation Functions. Now, let's concentrate on the following 4-point correlation function:

$$\langle V_{\mathbf{a}_+}(z)V_{\mathbf{a}_1}(z_1)V_{\mathbf{a}_2}(z_2)V_{\mathbf{a}_3}(z_3)\rangle = = G_{\mathbf{a}_+\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3}(z,z_1,z_2,z_3), \quad (4)$$

where the special vertex operator

$$V_{\mathbf{a}_+}(z) = e^{2\mathbf{a}_+\cdot\varphi}, \quad \mathbf{a}_+ = \left(-b, b/\sqrt{3}\right)$$

satisfies the null vector equation

$$[\Delta_+(5\Delta_+ + 1)W_{-2} - 12w_+ L_{-1}^2 + + 6w_+(\Delta_+ + 1)L_{-2}]V_{\mathbf{a}_+} = 0. \quad (5)$$

Taking into account the last equation and the explicit representation of the current W in terms of the field $\partial\varphi$ (see Ref. [8]), we find that the selected 4-point correlation function satisfies the differential equation

$$(\Delta_+ + 1)\frac{\partial^2}{\partial z^2}\langle V_{\mathbf{a}_+}(z)V_{\mathbf{a}_1}(z_1)V_{\mathbf{a}_2}(z_2)V_{\mathbf{a}_3}(z_3)\rangle - - 2\sum_{i=1}^{3}\left[\frac{\Delta_i+\delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right]\times \times \langle V_{\mathbf{a}_+}(z)V_{\mathbf{a}_1}(z_1)V_{\mathbf{a}_2}(z_2)V_{\mathbf{a}_3}(z_3)\rangle + + 4\sum_{i=1}^{3}\frac{A_i}{z-z_i}\langle V_{\mathbf{a}_+}\cdots\partial\varphi_1 V_{\mathbf{a}_i}\cdots\rangle + + 4\sum_{i=1}^{3}\frac{B_i}{z-z_i}\langle V_{\mathbf{a}_+}\cdots\partial\varphi_2 V_{\mathbf{a}_i}\cdots\rangle = 0, \quad (6)$$

where

$$\delta_i = -2\sqrt{2}i[2a_{+2}(a_{i2}^2 - a_{i1}^2) + 2a_{i2}(a_{+1}^2 - a_{+2}^2) +$$
$$+ a_{+2}a_{i1}(4a_{+1} - Q) - a_{+1}a_{i2}(4a_{i1} - Q)],$$
$$A_i = 2\sqrt{2}i(a_{+2}a_{i1} + a_{+1}a_{i2}),$$
$$B_i = 2\sqrt{2}i(a_{+1}a_{i1} - a_{+2}a_{i2}).$$

Moreover, for the special ratios

$$\frac{a_{i2}}{a_{i1}} = -\frac{a_{+2}}{a_{+1}} \pm \sqrt{1 + \left(\frac{a_{+2}}{a_{+1}}\right)^2} \qquad (7)$$

of the charges $\mathbf{a}_i$, equation (6) can be further reduced to the equation

$$(\Delta_+ + 1)\frac{\partial^2}{\partial z^2}\langle V_{\mathbf{a}_+}(z)V_{\mathbf{a}_1}(z_1)V_{\mathbf{a}_2}(z_2)V_{\mathbf{a}_3}(z_3)\rangle -$$
$$- 2\sum_{i=1}^3 \left[\frac{\Delta_i + \delta_i}{(z - z_i)^2} + \frac{1 + A}{(z - z_i)}\frac{\partial}{\partial z_i}\right] \times$$
$$\times \langle V_{\mathbf{a}_+}(z)V_{\mathbf{a}_1}(z_1)V_{\mathbf{a}_2}(z_2)V_{\mathbf{a}_3}(z_3)\rangle = 0, \qquad (8)$$

where $A = \pm 2\sqrt{2}i\sqrt{a_{+1}^2 + a_{+2}^2}$. It is well-known that in the case of the four-point functions, the partial differential equation (8), using the projective Ward identities [12], can be reduced to the Riemann ordinary differential equation

$$\left\{\frac{1}{2}(\Delta_+ + 1)\frac{d^2}{dz^2} + \sum_{i=1}^3 \left[\frac{1 + A}{z - z_i}\frac{d}{dz} - \frac{\Delta_i + \delta_i}{(z - z_i)^2}\right] + \right.$$
$$\left. + (1 + A)\sum_{i<j}^3 \frac{\Delta_+ + \Delta_{ij}}{(z - z_i)(z - z_j)}\right\} \times$$
$$\times \langle V_{\mathbf{a}_+}(z)V_{\mathbf{a}_1}(z_1)V_{\mathbf{a}_2}(z_2)V_{\mathbf{a}_3}(z_3)\rangle = 0, \qquad (9)$$

where $\Delta_{ij} = \Delta_i + \Delta_j - \Delta_k$, $(k \neq i, j)$, $(i, j, k = 1, 2, 3)$.

4. Functional Equations for Structure Constants. Now any four-point function can be explicitly decomposed in terms of the three-point function

$$G_{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4}(z, \bar{z}) =$$
$$= \langle V_{\mathbf{a}_1}(z_1, \bar{z}_1)V_{\mathbf{a}_2}(z_2, \bar{z}_2)V_{\mathbf{a}_3}(z_3, \bar{z}_3)V_{\mathbf{a}_4}(z_4, \bar{z}_4)\rangle =$$
$$= \sum_{\mathbf{a}} \mathbb{C}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{Q} - \mathbf{a})\mathbb{C}(\mathbf{a}, \mathbf{a}_3, \mathbf{a}_4) \times$$
$$\times \left| F_{\mathbf{a}}\begin{pmatrix} \mathbf{a}_1\mathbf{a}_2 \\ \mathbf{a}_3\mathbf{a}_4 \end{pmatrix}(z, \bar{z})\right|^2. \qquad (10)$$

Conformal invariance allows us to set $z_1 = 0$, $z_2 = z$, $z_3 = 1$, $z_4 = \infty$. As a consequence, the crossing symmetry condition is written as

$$G_{\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4}(z, \bar{z}) = G_{\mathbf{a}_1\mathbf{a}_4\mathbf{a}_2\mathbf{a}_3}(1 - z, 1 - \bar{z}) =$$
$$= z^{-2\Delta_2}\bar{z}^{-2\bar{\Delta}_2}G_{\mathbf{a}_1\mathbf{a}_3\mathbf{a}_2\mathbf{a}_4}(1/z, 1/\bar{z}).$$

To discover additional information about the structure constants of the FTFT, we will use technique suggested in Ref. [13]. So, let's assume that $\mathbf{a}_2 = \mathbf{a}_+$, i.e. the correlation function (10) includes the degenerate field $V_{\mathbf{a}_+}$. Then the charges of the intermediate channel will take the following values [7]

$$(a_{11} + a_{+1}, a_{12} + a_{+2}),$$
$$(a_{11} - a_{+1}, a_{12} + a_{+2}), \quad (a_{11}, a_{12} - 2a_{+2}). \qquad (11)$$

This implies the following "fusion rules"

$$V_{\mathbf{a}_+}V_{\mathbf{a}} = [V_{a_1 + a_{+1}, a_2 + a_{+2}}] +$$
$$+ [V_{a_1 - a_{+1}, a_2 + a_{+2}}] + [V_{a_1, a_2 - 2a_{+2}}].$$

It is more convenient to introduce the following "parametrization" of the intermediate charge (11)

$$\mathbf{a}(s) = (a_{11} + sa_{+1}, a_{12} + (3s^2 - 2)a_{+2}),$$
$$s = 0, \ \pm 1.$$

Using this parametrization, we can rewrite (10) as follows:

$$G_{\mathbf{a}_1\mathbf{a}_+\mathbf{a}_3\mathbf{a}_4}(z, \bar{z}) =$$
$$= \sum_{s=0, \pm 1} \mathbb{C}(\mathbf{a}_1, \mathbf{a}_+, \mathbf{Q} - \mathbf{a}(s))\mathbb{C}(\mathbf{a}(s), \mathbf{a}_3, \mathbf{a}_4) \times$$
$$\times \left| F_s\begin{pmatrix} \mathbf{a}_1\mathbf{a}_+ \\ \mathbf{a}_3\mathbf{a}_4 \end{pmatrix}(z, \bar{z})\right|^2. \qquad (12)$$

In this notation the crossing symmetry relation for $G_{\mathbf{a}_1\mathbf{a}_+\mathbf{a}_3\mathbf{a}_4}(z, \bar{z})$ is

$$\sum_{s=0,\pm 1} \mathbb{C}_s(\mathbf{a}_1)\mathbb{C}(\mathbf{a}(s), \mathbf{a}_3, \mathbf{a}_4) \times$$
$$\times \left| F_s\begin{pmatrix} \mathbf{a}_1\mathbf{a}_+ \\ \mathbf{a}_3\mathbf{a}_4 \end{pmatrix}(z, \bar{z})\right|^2 =$$
$$= |z|^{-4\Delta_2} \sum_{p=0,\pm 1} \mathbb{C}_p(\mathbf{a}_4)\mathbb{C}(\mathbf{a}(p), \mathbf{a}_3, \mathbf{a}_1) \times$$
$$\times \left| F_p\begin{pmatrix} \mathbf{a}_4\mathbf{a}_+ \\ \mathbf{a}_3\mathbf{a}_1 \end{pmatrix}(1/z, 1/\bar{z})\right|^2, \qquad (13)$$

where we have denoted

$$\mathbb{C}(\mathbf{a}_1, \mathbf{a}_+, \mathbf{Q} - \mathbf{a}(s)) = \mathbb{C}_s(\mathbf{a}_1)$$

and

$$\mathbb{C}(\mathbf{a}_4, \mathbf{a}_+, \mathbf{Q} - \mathbf{a}(p)) = \mathbb{C}_p(\mathbf{a}_4).$$

It follows from (9) that the conformal block must satisfy the following relation

$$F_s\begin{pmatrix}\mathbf{a}_1\mathbf{a}_+\\ \mathbf{a}_3\mathbf{a}_4\end{pmatrix}(z,\bar z)=$$
$$=z^{-2\Delta_+}\sum_{p=0,\pm1}M_{ps}F_p\begin{pmatrix}\mathbf{a}_4\mathbf{a}_+\\ \mathbf{a}_3\mathbf{a}_1\end{pmatrix}(1/z,1/\bar z),\quad(14)$$

where M_{ps} is a matrix that is determined by the monodromy properties of the differential equation equation (9) or, alternatively, can be determined by the method developed in Ref. [14]. The exact analitical expression of the matrix M_{ps} can be found in the number of the papers [8, 15, 14]. We will not write down these expressions for the reason of the limited frame of the paper.

Substituting (14) into (13), we find the following functional equations for the A_2 FTFT structure constants:

$$\begin{aligned}&\sum_{s=0,\pm1}\mathbb{C}_s(\mathbf{a}_1)\mathbb{C}(\mathbf{a}(s),\mathbf{a}_3,\mathbf{a}_4)M_{s,0}\overline{M}_{s,1}=0,\\&\sum_{s=0,\pm1}\mathbb{C}_s(\mathbf{a}_1)\mathbb{C}(\mathbf{a}(s),\mathbf{a}_3,\mathbf{a}_4)M_{s,0}\overline{M}_{s,-1}=0,\\&\sum_{s=0,\pm1}\mathbb{C}_s(\mathbf{a}_1)\mathbb{C}(\mathbf{a}(s),\mathbf{a}_3,\mathbf{a}_4)M_{s,1}\overline{M}_{s,-1}=0,\end{aligned}\quad(15)$$

provided $\mathbf{a}_1,\mathbf{a}_3,\mathbf{a}_4$ satisfy the constraint (7).

It is important to notice that Eq. (5), admits additional solutions besides $\mathbf{a}_+$. In particular, $\mathbf{a}^+=(-b,-b\sqrt{3})$, $\mathbf{a}_-=(-1/b,1/b\sqrt{3})$, $\mathbf{a}^-=(-1/b,-1/b\sqrt{3})$ are all solutions of (5). Therefore the set of Eqs. (15) should be complemented by a similar set of equations obtained for the special case $\mathbf{a}^+$ and then add for each equation its 'dual equation' using the subsitutions $b\to1/b$ and $\mu\to\tilde\mu$. The parameter $\tilde\mu$ is defined by duality relations [16]

$$\pi\mu\gamma\left(\frac{\mathbf{e}_i^2b^2}{2}\right)=\pi\tilde\mu\gamma\left(\frac{2}{\mathbf{e}_i^2b^2}\right)^{\mathbf{e}_i^2b^2/2},$$

where $\gamma(x)=\Gamma(x)/\Gamma(1-x)$.

In principale, in terms of the special function "Upsilon" [10, 11], the complete set of the algebraic equations derived above for the special cases $\mathbf{a}_+,\mathbf{a}^+,\mathbf{a}_-,\mathbf{a}^-$ allows the computation of all structure constants for the A_2 FTFT. We postpone the difficult problems of the exact determination of the structure constants and proof of the uniqueness of the solution for future studies.

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