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Reentrant violation of special relativity in the low-energy corner

G. E. Volovik

Low Temperature Laboratory, Helsinki University of Technology
FIN-02015 HUT, Finland

L. D. Landau Institute for Theoretical Physics, 117334 Moscow, Russia

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In the effective relativistic quantum field theories the energy region, where the special relativity holds, can be sandwiched from both the high and low energies sides by domains where the special relativity is violated. An example is provided by ^{3}He-A where the relativistic quantum field theory emerges as the effective theory. The reentrant violation of the special relativity in the ultralow energy corner is accompanied by the redistribution of the momentum-space topological charges between the fermionic flavors. At this ultralow energy an exotic massless fermion with the topological charge $N_3 = 2$ arises, whose energy spectrum mixes the classical and relativistic behavior. This effect can lead to neutrino oscillations if neutrino flavors are still massless at this energy scale.

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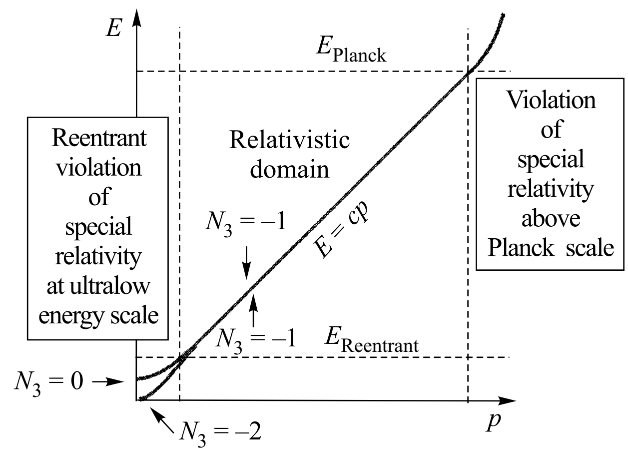
Introduction. The condensed matter analogy supports an idea that the special and general relativity might be emergent properties of quantum vacuum, which arise gradually in the low energy corner [1–3]. If this is true one can expect that the Lorentz invariance of our low-energy world is violated at high energy. The condensed matter provides examples of how this violation can occur. Here we demonstrate one generic example which is realized in superfluid ^{3}He-A, where the effective special relativity and gravity do arise in the low energy corner together with the chiral fermions and effective gauge fields [3]. It suggests that if the effective Lorentz invariance is violated in the extreme limit of "Planckian" scale, it becomes violated also in the opposite extreme limit of ultralow energy. The energy scale where the reentrant violation of the special relativity occurs is also dictated by the "trans-Planckian" physics. If there are still fermions which remain chiral and massless when approaching this ultralow energy scale, the reentrant violation of the Lorentz invariance leads to the crucial reconstruction of their energy spectrum. Thus the trans-Planckian physics can be probed in the limit of low energies.

In this example two flavors of chiral left-handed fermions are hybridized producing one massive fermion with the relativistic spectrum $E^2 = c^2p^2 + m^2c^4$ and one exotic gapless fermion whose spectrum mixes the classical and relativistic behavior: $E^2 = c^2p_\|^2 + (p_\perp^2/2m)^2$ (see Figure). Such energy spectrum is the consequence of the nontrivial momentum space topology. The hybridization of fermions due to violation of special relativity can provide the scenario for neutrino oscillations, similar to that discussed in Ref.[4] where also the violation of special relativity was considered, but in terms of different maximum attainable velocity c for different species of neutrino.



Low energy memory of the high energy nonsymmetric physics

Momentum space topology and discrete symmetry between the fermions. The special relativity (and also the general relativity with the effective gravitational field being one of the collective modes of the fermionic vacuum) naturally arises in such Fermi superfluids whose fermionic quasiparticle spectrum contains topologically nontrivial point nodes in momentum space. Examples are the superfluid phases of ^{3}He: ^{3}He-A and the planar state. In the low energy limit, i.e. in the vicinity of a given topologically stable point node (the Fermi point), fermionic quasiparticles behave as chiral fermions with the masless spectrum obeying the relativistic-like equation

$$g^{\mu\nu}_{(a)}(p_\mu - p_{\mu(a)})(p_\nu - p_{\nu(a)}) = 0. \quad (1)$$

Here $p_{\mu(a)}$ (the position of the node in the spectrum of the a-th quasiparticle) and $g^{\mu\nu}_{(a)}$ are dynamic variables describing the collective bosonic degrees of freedom of the vacuum. They play the role of the gauge and gravity fields respectively.

In each of the two phases of superfluid ^{3}He (^{3}He-A and the planar state) there is a symmetry, which connects all the low-energy fermionic species. As a result, the effective metric $g_{\mu\nu}$ is the same for all fermions (at least in equilibrium) which means that all of them have the same "speed of light" (i.e. the same maximum attainable speed). Moreover, for the "perfect" fermionic system (for which the Lagrangian for the collective bosonic modes is obtained by the integration over the fermions in the vicinity of the Fermi points, see [3]) the bosonic fields are governed by the same effective metric $g_{\mu\nu}$ as fermions and thus have the same speed of light. Thus the nontrivial momentum-space topology and the symmetry between the fermions are two ingredients for establishing the special relativity in the low energy corner of the effective theory.

The massless (gapless) character of the fermionic spectrum in the system with Fermi points is protected by the nonzero value of the topological invariant of the ground state, which is expressed as the integral over the Green's function in the 4D momentum-frequency space:

$$N_3 = \frac{1}{24\pi^2} e_{\mu\nu\lambda\gamma}\, \mathbf{tr} \int_\sigma dS^\gamma\, \mathcal{G}\partial_{p_\mu}\mathcal{G}^{-1}\mathcal{G}\partial_{p_\nu}\mathcal{G}^{-1}\mathcal{G}\partial_{p_\lambda}\mathcal{G}^{-1}. \quad (2)$$

The integral here is over the surface σ embracing the Fermi point $p_{\mu(a)} = (\mathbf{p}_a, 0)$ in the 4D momentum space $p_\mu = (\mathbf{p}, p_0)$ and p_0 is the energy (frequency) along the imaginary axis; **tr** is the trace over the fermionic indices. If the value of the topological charge of the Fermi point is $N_3 = +1$ or $N_3 = -1$, then in the vicinity of this point the quasiparticle is a massless fermion, whose Green's function after the proper rescaling and shifting the position of the Fermi point has the following form

$$\mathcal{G} = (ip_0 - \mathcal{H})^{-1} \quad , \quad \mathcal{H} = N_3 c\sigma \cdot \mathbf{p}. \quad (3)$$

This is the Green's function of the left-handed (if $N_3 = -1$) or right-handed (if $N_3 = +1$) chiral fermion. Thus the topologically nontrivial momentum-space topology automatically produces the "relativistic" chiral fermions as the low-energy quasiparticles, if $N_3 = \pm 1$.

In the gapless superfluid phases of ^{3}He the topological invariants of each of two Fermi points are different from ± 1: they are $N_3 = \pm 2$ in ^{3}He-A and $N_3 = 0$ in the planar state. In both phases, however, there is a discrete symmetry between the fermionic species, which leads to the equal (in ^{3}He-A) or opposite (in the planar state) distribution of the topological charges between fermions: $N_3 = +2 \to +1 + 1$ and $N_3 = -2 \to -1 - 1$ in ^{3}He-A, and $N_3 = 0 \to +1 - 1$ in the planar state. As a result each gapless fermion has unitary charge, $N_3 = -1$ or $N_3 = +1$, and thus all gapless quasiparticles are relativistic in the low-energy corner. This again shows the importance of the discrete or continuous symmetry between the fermions for the special relativity to hold in the low energy corner. On the discrete symmetry which provides the unitary charges $N_3 = -1$ or $N_3 = +1$ for the chiral fermions in the Standard Model see Ref. [3].

Condensed mater scenario of the reentrant violation of special relativity. In ^{3}He-A, the global symmetry $SO(3)_S \times SO(3)_L \times U(1)_N$ of the normal ^{3}He is broken to the little group $U(1)_{L_z - N/2} \times U(1)_{S_z}$ whose generators are $L_z - N/2$ and S_z. Here $SO(3)_S$, $SO(3)_L$ and $U(1)_N$ are correspondingly spin rotation group, group of orthogonal coordinate transformations and the global $U(1)$ group responsible for the conservation of the global charge – the number of ^{3}He atoms. The corresponding 3×3 order parameter matrix $A_{\mu i}$, which transforms as a vector under spin rotations $SO(3)_S$ (the first index) and as a vector under orbital rotations $SO(3)_L$ (the second index), is

$$A_{\mu i} = \Delta_0 \hat{z}_\mu \,(\hat{x}_i + i\hat{y}_i)\,. \quad (4)$$

Fermionic quasiparticles living in the vacuum with such order parameter have two point nodes in the spectrum. In the vicinity of the Fermi point at $\mathbf{p} = (0, 0, p_F)$, which has with the topological charge $N_3 = -2$, these quasiparticles correspond to two chiral left-handed "relativistic" fermions described by the following Bogliubov – Nambu Hamiltonian

$$\mathcal{H}_{\mathrm{A-phase}} = c_\parallel (p_z - p_F)\check{\tau}_3 + c_\perp \sigma_z (\check{\tau}_1 p_x - \check{\tau}_2 p_y). \quad (5)$$

Here the "speeds of light" propagating along and transverse to the axis z are correspondingly $c_\parallel = v_F$ and $c_\perp = \Delta_0/p_F$; v_F and p_F are the Fermi velocity and Fermi momentum in the normal ^{3}He; Δ_0 is the amplitude of the gap; σ_z is the Pauli matrix for the nuclear spin of ^{3}He atom; $\check{\tau}_i$ are the Pauli matrices for the Bogoliubov – Nambu spin in the particle-hole space.

Two projections of atomic spin $(1/2)\sigma_z = \pm 1/2$ can be considered as two fermionic flavors. Each of two fermions has $N_3 = -1$, that is why both fermions are chiral (left-handed) and massless, with the energy spectrum

$$E^2 = c_{\parallel}^2 \tilde{p}_z^2 + c_{\perp}^2 p_{\perp}^2 \ , \quad N_3 = -1, \tag{6}$$

where $\tilde{p}_z = p_z - p_F$. The symmetry, which couples the two flavors and forces them to have identical topological charge $N_3 = -1$ and identical "relativistic" spectrum, is the discrete symmetry U_2 of the order parameter in Eq.(4). U_2 is the combined symmetry: it is the element of the $SO(3)_S$ group, the π rotation of spins about, say, axis x, which is supplemented by the gauge rotation $e^{i\pi}$ from the $U(1)_N$ group: $U_2 = C_\pi^x e^{i\pi}$. It is the same discrete symmetry which gives rise to the Alice string (the half-quantum vortex, for review see [5]). Such symmetry came from the fundamental level well above the first Planck scale $E_{\rm Planck} = \Delta_0^2/v_F p_F$ [3] at which the spectrum becomes nonlinear and thus the Lorentz invariance is violated.

However, this discrete U_2 symmetry is not exact in ^{3}He even on the fundamental level. This is because of the tiny spin-orbit interaction, which slightly violates the symmetry $SO(3)_S$ under the separate rotations of spin space. Since the U_2 symmetry was instrumental for the establishing of the special relativity in the low energy corner, its violation must lead to violation of the Lorentz invariance and also to mixing of the two fermionic flavors at the very low energy determined by the tiny spin-orbit coupling.

Let us consider how all this happens in ^{3}He-A. Due to the spin-orbit coupling, the $U(1)_{L_z - N/2} \times U(1)_{S_z}$ symmetry of ^{3}He-A is not exact. The remaining exact symmetry is the combined symmetry constructed from the sum of two generators: $U(1)_{J_z - N/2}$, where $J_z = S_z + L_z$ is the generator of the simultaneous rotations of spins and orbital degrees of freedom. As a result the order parameter in Eq.(4) acquires a small correction consistent with the $U(1)_{J_z - N/2}$ symmetry:

$$A_{\mu\, i} = \Delta_0 \hat{z}_\mu \left(\hat{x}_i + i\hat{y}_i\right) + \alpha\Delta_0 \ \left(\hat{x}_\mu + i\hat{y}_\mu\right) \hat{z}_i. \tag{7}$$

The first term corresponds to Cooper pair state with $L_z = 1/2$ per atom and $S_z = 0$, while the second one is a small admixture of state with $S_z = 1/2$ and $L_z = 0$. Both components have $J_z = 1/2$ and thus must be present in the order parameter. Due to the second term the order parameter is not symmetric under the U_2 operation. The small parameter $\alpha \sim \xi^2/\xi_D^2 \sim 10^{-5}$ is the relative strength of the spin-orbit coupling, where $\xi \sim 10^{-6} - 10^{-5}$ cm is the superfluid coherence length and $\xi_D \sim 10^{-3}$ cm is the so called dipole length characterizing the spin-orbit coupling.

The Bogoliubov – Nambu Hamiltonian for fermionic quasiparticles in such vacuum is now modified as compared with that in the pure vacuum state with $L_z = 1/2$ and $S_z = 0$ in Eq.(5):

$$\begin{aligned} \mathcal{H}_{\rm A-phase} = c_{\parallel} \tilde{p}_z \check{\tau}_3 + c_{\perp}\sigma_z (\check{\tau}_1 p_x - \check{\tau}_2 p_y) + \\ + \alpha c_{\perp} p_z (\sigma_x \check{\tau}_1 - \sigma_y \check{\tau}_2). \end{aligned} \tag{8}$$

Diagonalization of this Hamiltonian shows that the small correction due to spin-orbit coupling gives rise to the following splitting of the energy spectrum

$$E_{\pm}^2 = c_{\parallel}^2 \tilde{p}_z^2 + c_{\perp}^2 \left(\alpha |p_z| \pm \sqrt{\alpha^2 p_z^2 + p_{\perp}^2}\right)^2 . \tag{9}$$

Near the Fermi point one can put $\alpha|p_z| = \alpha p_F$. The + and - branches give the gapped and gapless spectra correspondingly. For $p_\perp \ll \alpha p_F$ one has

$$E_{+}^2 \approx c_{\parallel}^2 \tilde{p}_z^2 + \tilde{c}_{\perp}^2 p_{\perp}^2 + \tilde{m}^2 \tilde{c}_{\perp}^4, \tag{10}$$

$$E_{-}^2 \approx c_{\parallel}^2 \tilde{p}_z^2 + \frac{p_\perp^4}{4\tilde{m}^2}, \tag{11}$$

$$\tilde{c}_\perp = \sqrt{2} c_\perp, \ \tilde{m} = \alpha \ \frac{p_F}{c_\perp}, \ p_\perp \ll \tilde{m} c_\perp. \tag{12}$$

In this ultra-low energy corner the gapped branch of the spectrum in Eq.(10) is relativistic, though with different speed of light than in the intermediate regime of Eq.(6). The gapless branch in Eq.(11) is relativistic in one direction $E = c_{\parallel}|\tilde{p}_z|$, and is classical, $E = p_\perp^2/2\tilde{m}$, for the motion in the transverse direction.

Momentum space topology of exotic fermion. What is important here is that such splitting of the spectrum is generic and thus can occur in other effective field theories such as Standard Model. This is because of the topological properties of the spectrum: the mixing of the two fermionic flavors occurs with the redistribution of the topological charge between the two fermions. In the relativistic domain each of two fermions has the topological charge $N_3 = -1$. It is easy to check that in the ultra low energy corner it is not the case. While the total topological charge of the Fermi point $N_3 = -2$ must be conserved, it is now redistributed between the fermions in the following manner: the massive fermion (with the energy E_+) acquires the trivial topological charge $N_3 = 0$ (that is why it becomes massive), while another one (with the energy E_-) has double topological charge $N_3 = -2$ (see Figure). It is important that the topological charge $N_3 = -2$ describes single fermionic species: it cannot split into two fermions with $N_3 = -1$ each. This exotic fermion with $N_3 = -2$ is gapless because of the nonzero value of the topological charge, but the energy spectrum of such fermion is nonlinear. Such spectrum cannot be described in the relativistic language.

In the same way as the $N_3 = \pm 1$ fermions are necessarily relativistic and chiral in the low-energy corner, the fermions with higher $|N_3|$ are necessarily nonrelativistic. The momentum space topology which induces the special relativity if $|N_3| = 1$, becomes incompatible with the relativistic invariance if $|N_3| > 1$, and the latter is obligatory violated. Properties of the fermionic systems with multiple zeroes, $|N_3| > 1$, including the axial anomaly in its non-relativistic version were discussed in Ref.[6].

The energy scale at which the splitting of the energy spectrum occurs is $E_{\rm Reentrant} = \alpha\Delta_0$ which is much less than the first Planck level in ^{3}He-A, $E_{\rm Planck} = \Delta_0^2/v_F p_F$. Thus the relativistic region for the ^{3}He-A fermions, $E_{\rm Reentrant} \ll E \ll E_{\rm Planck}$, is sandwiched from both the high and low energies by the nonrelativistic regions.

Discussion. The above example of ^{3}He-A shows that the discrete symmetry between the fermions together with the momentum-space topology guarantee that massless fermions obey the special relativity in the low-energy corner. If the discrete symmetry is approximate, then in the ultralow energy corner the redistribution of the momentum-space topological charges occurs between the fermions with appearance of the higher topological charge $|N_3| > 1$. This topological transition leads to the strong modification of the energy spectrum which becomes essentially non-relativistic.

In principle, such topological transition with appearance of the exotic fermions with $N_3 = \pm 2$ can occur in the relativistic quantum field theories too, if these theories are effective. In the effective theory the Lorentz invariance (and thus the special relativity) appears in the low-energy corner as an emergent phenomenon, while it can be violated at high energy approaching the Planck scale. At low energy the fermions are chiral and relativistic, if there is a symmetry between the flavors. If such symmetry is violated, either spontaneously or due to the fundamental physics above the Planck scale, then in the extreme low energy limit, when the asymmetry between the fermionic flavors becomes important, the system remembers its high-energy nonrelativistic origin. The rearrangement of the topological charges N_3 between the fermionic species occurs and the special relativity disappears again.

This scenario can be applied to the massless neutrinos. The violation of the horizontal symmetry between the left-handed neutrino flavors can lead to the violation of Lorentz invariance at the very low energy. If the neutrinos remain massless at this ultralow energy scale, then below this scale the two flavors (say, electronic and muonic left-handed neutrinos each with $N_3 = -1$) hybridize and produce the $N_3 = 0$ fermion with the gap and the exotic gapless $N_3 = -2$ fermion with the essentially nonlinear non-relativistic spectrum. This is another example of violation of the special relativity, which can also give rise to the neutrino oscillations. The previously considered effect of the violation of the special relativity on neutrino oscillationis was related to the different speeds of light for different neutrino flavors [4] (the related effect is the violation of the weak equivalence principle: different flavors are differently coupled to gravity [7]).

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