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Focusing of nonlinear wave groups in deep water

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The freak wave phenomenon in the ocean is explained by the nonlinear dynamics of phase modulated wave trains. It is shown that the preliminary quadratic phase modulation of wave packets leads to the significant amplification of the "usual" modulation (Benjamin-Feir) instability. Physically, the phase modulation of water waves may be due to a variable wind in storm areas. The well-known breather solutions of the cubic Schrodinger equation appear on the final stage of the nonlinear dynamics of the wave packets, when the phase modulation becomes more uniform.

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The increase of a number of reported damages of ships and offshore platforms is explained very often by the freak wave appearance on the sea surface [1, 2]. Several physical mechanisms of the freak wave phenomenon are discussed. First of all, the water wave interaction with opposite current is considered as a mechanism of wave amplification due to the blocking of water waves on the current. This phenomenon is investigated within the framework of the wave action balance equation and the variable-coefficient nonlinear Schrödinger equation [2 – 4]. The second mechanism of wave amplification is related with the formation of caustics in the wave field on random currents [5]. These theories are used to explain the freak wave formation due to the Agulhas current off the south-east coast of South Africa. Many observations of abnormal waves had been done in areas with no strong currents. For such areas the opinion that the nonlinearity of surface waves in deep water can produce the giant wave by itself becomes very popular [6, 7]. The theory is based on the modulation instability of water waves (see the review [8]), and existence of breather-like solutions of the nonlinear Schrödinger equation [7, 9, 10 – 13]. The amplitude of breathers can exceed the amplitude of unperturbed non-modulated wave trains more than twice (remind that it is the formal definition of a freak wave). The nonlinear Schrödinger equation is a simplified model of real wind waves, more sophisticated models are applied too (Zakharov equation, Dysthe equation, etc). In [6] a freak wave formation due to modulation instability computed by a numerical model of the full-nonlinear hydrodynamic potential equations was compared with cubic nonlinear Schrödinger equation and found a good agreement if the steepness of waves is not too large. According to these theories freak waves exist during the characteristic time scale of the modulation instability and may propagate on relatively large distance. Meanwhile the event descriptions emphasize the very short-lived character of the freak wave. In our opinion, the mechanism of the focusing of water wave packets related to the phase (frequency) modulation should play a significant role in the short-lived freak wave formation. This mechanism is well-known in the linear theory of dispersive waves [14] and may occur for specific meteorological conditions. For instance, the increase of wind speed generates during the early stages wave packets with low group velocities and later wave packets with larger group velocities. The result of the propagation process is the formation of an impulse of very large amplitude which is due to the superposition of many spectral packets. Analytic solutions, proving this linear focusing mechanism, are presented in [15]. In laboratory tanks the phenomenon of significant wave focusing was reported [16, 17] for a wide variation of the wavelength/depth ratio for deep water and shallow water as well. Recently, [18] showed that the mechanism of wave focusing can be applied in the weakly nonlinear theory of shallow water [2)] (Korteweg – de Vries model) and suggested the way to find possible forms of wave trains moving towards the freak wave, including random background of wind waves. Owing to

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[2)]The process of focusing of phase-modulated impulses in nonlinear media is also known and applied in optics [19, 20].

the absence of the modulation instability for shallow water, the wave focusing mechanism seems to be major in shallow water.

The present paper deals with the freak wave formation in deep water due to the focusing of nonlinear wave packets with phase modulation. This mechanism is compared with the possible generation of giant waves (breathers) due to the "usual" modulation instability of water waves. The main result of the paper is that the frequency modulation of a nonlinear wave field can lead to larger amplification of the freak wave than the amplitude modulation usually considered by previous authors.

The simple model of weakly nonlinear deep-water wave packets is the famous cubic Schrödinger equation

$$i\frac{\partial A}{\partial t} + \frac{\partial^2 A}{\partial x^2} + 2|A|^2 A = 0, \qquad (1)$$

where in dimensionless variables A is proportional to the wave steepness, $A = \sqrt{2}k_0 a$, a is the amplitude of the surface elevation, k_0 and ω_0 are the carrier wave number and frequency respectively, $x = 2k_0 x' - \omega_0 t'$, $t = \omega_0 t'/2$, x' and t' are coordinate and time. Due to the invariance transformation of (1), $t \to -t$, $i \to -i$, the simplest algorithm to find the nonlinear wave packets moving towards the giant wave can be suggested: take the expected form of the freak wave as an initial condition for (1), and consider the resulting field as the initial condition that gives the freak wave under the invariant transformation. The solution of the Cauchy problem for the nonlinear Schrödinger equation is known by using the inverse scattering method, see the pioneering paper [21]. In general, the scattering data include both, continuous and discrete spectra. The continuous spectrum corresponds to the dispersive wave packets. In the case of no discrete spectrum, the solution of (1) tends to the phase-modulated wave for large times

$$A(x,t) = \frac{Q}{\sqrt{t}} \exp\left[i\left(\frac{x^2}{4t} + 2Q^2 \ln t + \theta\right)\right], \qquad (2)$$

where Q and θ are functions of (x/t) [22]. When Q and θ are real constants, (2) gives an exact solution of (1), which is almost the same as the self-similar solution of the linear version of (1). The difference is in the logarithmic term of the phase. Replacing t on $T - t$ and i on $-i$, the solution (2) describes the transformation of the initial frequency modulated wave into the delta-function formally representing the freak wave. Therefore, the mechanism of wave focusing is valid in linear theory and in nonlinear theory as well, but the nonlinearity influences the optimal phase (wavenumber) distribution of individual waves due to the logarithmic term that depends on wave intensity.

Each discrete eigenvalue λ of the spectrum corresponds to an isolated soliton with amplitude $2a$ and speed $4b$, where $a = \mathrm{Re}(\lambda)$ and $b = \mathrm{Im}(\lambda)$ [21]. The number of the discrete eigenvalues depends on the form and energy of the initial disturbance. If the expected freak wave has the form of an isolated crest with vanishing tails at infinity and no phase modulation, all discrete eigenvalues are real and correspond to the "standing" solitary waves. Due to interaction between them, the resulting wave field is unsteady and shows the complex picture of the oscillating impulses of non-constant form. In particular, the two-soliton solution (bi-soliton) describes a wave which breathes with the period $T = \pi/2|a_2^2 - a_1^2|$, its peak value is $A_{max} = 2(a_1 + a_2)$ where $a_1 = \mathrm{Re}(\lambda_1)$ and $a_2 = \mathrm{Re}(\lambda_2)$. The discrete spectrum is found for several forms of initial disturbances, in particular, [23] considered the profile $A_{fr}(x) = A_p \mathrm{sech}(x/L)$ (index *fr* refers to the freak wave). Eigenvalues are positive and equal to

$$\lambda_n L = \frac{M}{\pi} - n + \frac{1}{2}, \quad n = 1, 2, ..., N, \qquad (3)$$

where the number of eigenvalues is

$$N = E\left[\frac{M}{\pi} + \frac{1}{2}\right], \qquad (4)$$

E is the integer function and M is the mass of the freak wave, $M = \pi A_p L$.

It is important to note that, if $M < \pi/2$, there is no soliton generation, and this case can be considered as "quasi-linear". The wave evolves like the self-similar solution (2): at first, the wave focuses on short freak wave, and then disperses. The one soliton forms, if $\pi/2 < M < 3\pi/2$, and its amplitude will vary from 0 to $2A_p$. In the latter case, the soliton amplitude exceeds the amplitude of the initial disturbance. If we consider such a wave group (one soliton with amplitude $2A_p$ and the dispersive tail) as an initial condition, it will transform into the sech-disturbance, but it has no specificity of the expected freak wave (its amplitude should be large on the background of other waves). If we introduce the formal definition of the freak wave (its amplitude exceeds at least twice the background amplitude), it means that the amplitude of the freak wave should satisfy the following condition

$$M < 2\pi/3. \qquad (5)$$

Therefore, the freak wave generated from the bounded wave group is a weakly nonlinear wave. Such a wave can be generated by dispersive wave packets only, if $M < \pi/2$, or by dispersive wave packets plus the single soliton, if $\pi/2 < M < 2\pi/3$. The interaction between solitons only cannot generates the large-amplitude impulse; it will be comparable with solitons in amplitude. If we consider other profiles of the expected freak wave, different from the sech-function, integer constants in (3), (4) will change (see, for instance, [24]), but in the same order of magnitude. If we consider the initial impulse with the quadratic phase modulation like $\exp(iqx^2)$, the discrete eigenvalues will increase with q [25]. This result is obvious, because such a disturbance will transform, first, into an impulse with no phase modulation due to the wave focusing, and this large impulse leads to the large eigenvalues. Therefore, the form of the expected freak wave can be taken with no phase modulation and details of the waveform have no principal significance for the understanding of the wave focusing phenomenon in the nonlinear medium. Described above the mechanism of freak wave formation from the bounded wave packets is the same as for shallow water [18].

The mechanism of the localized wave formation (solitons or breathers) from the preliminary plane wave due to the modulation instability is studied since almost 20 years. Several nonlinear structures can be considered as models for the freak wave, its peak amplitude exceeds more than twice the unperturbed value. First of all, there is the Ma-breather [7, 9]

$$A(x,t) = \frac{\cos(\Omega t - 2i\varphi) - \cosh(\varphi)\cosh(px)}{\cos(\Omega t) - \cosh(\varphi)\cosh(px)}\exp(2it), \tag{6}$$

where $p = 2\sinh(\varphi)$, $\Omega = 2\sinh(2\varphi)$, and φ is an arbitrary positive constant. This wave tends to the unperturbed plane wave of unit amplitude for $|x| \to \infty$, and its amplitude is time periodic with the frequency Ω. The peak value of the breather exceeds twice and more the unperturbed value for $0 < \varphi < 0.96$. It is important to emphasize that the freak wave phenomenon has a periodic character in this model.

Another solution, called "homoclinic orbit" was found in [11, 13] and may be expressed by (6) if to change φ to $-i\varphi$, p to $-ip$ and Ω to $-i\Omega$. This wave is periodic in space, and tends to the unperturbed plane wave when $|t| \to \infty$. The maximal peak value of the impulse is less than 3. This "homoclinic orbit" can be considered also as a model for the freak wave for $\varphi > \pi/3$. It is important to mention that freak waves in this model should appear simultaneously in many spatial points.

Both breather solutions considered above for $\varphi = 0$ transform into the "algebraic" breather [10] with an increase of the peak value three times the value of the unperturbed wave amplitude. During the moment of maximal amplification ($t = 0$), the freak wave represents the large crest above the unperturbed plane wave ($|x| < 1/2$), and two depressions up to zero. The mass of this positive crest only is

$$M = \int_{-1/2}^{+1/2} |A(x,0)|dx = 1 + 4\mathrm{atanh}(1/2) \approx 2.9. \tag{7}$$

Thus, the breather solution provides a model for the freak wave with mass greater than for "pure focusing" regime, see (5). This is the main "kinematic" difference between "focused" and "nonlinear" freak waves. Physically, this difference can be clarified as follows. The "focused" freak wave is formed by the superposition of many spectral components, and the number of spectral components, or the effective spectrum width, K_0 should be large to provide the narrow crest. The "focused" freak wave is a very weakly nonlinear dispersive wave, and it should be narrow if dispersion prevails over nonlinearity. Its time of existence can be very small. The "nonlinear" freak wave is due to the modulation instability. As it is well-known (see, for instance, [8]), the width of the unstable spectral domain is $K_{BF} = 2A_0$, and the characteristic time-scale of the instability is $T_{BF} \sim 1/2A_0^2$, where A_0 is the amplitude of the unperturbed plane wave. For "nonlinear" freak wave, dispersion and nonlinearity are of the same order. Its time of existence is the characteristic time of the modulation instability.

For small wave amplitudes the width of the modulation instability is narrow and the "focusing" mechanism should dominate. With the increase of the unperturbed wave amplitudes, K_{BF} will become comparable to K_0, and the spectral components will contribute in both processes of the formation of the freak wave. In the case of no specific phase modulation of the wave packet, the "nonlinear" mechanism of the freak wave phenomenon should dominate the general dynamics of the wave field. But if the specific order of the "draw up" of the spectral components is organized (for instance due to the wind action), the phase modulation can cardinally change the modulation instability. The important role of frequency modulation on the modulation instability was emphasized by [26]. He pointed out that the small focusing effects may have a destabilizing effect under certain conditions. Nevertheless it seems that the nonlinear stage of the modulation instability for frequency modulated wave packets was not yet investigated previously in literature. The effect of the quadratic phase modulation on the nonlinear evolution of the modulation instability is considered here numerically.

The nonlinear Schrödinger equation is solved by using a pseudo-spectral method in a periodic domain of

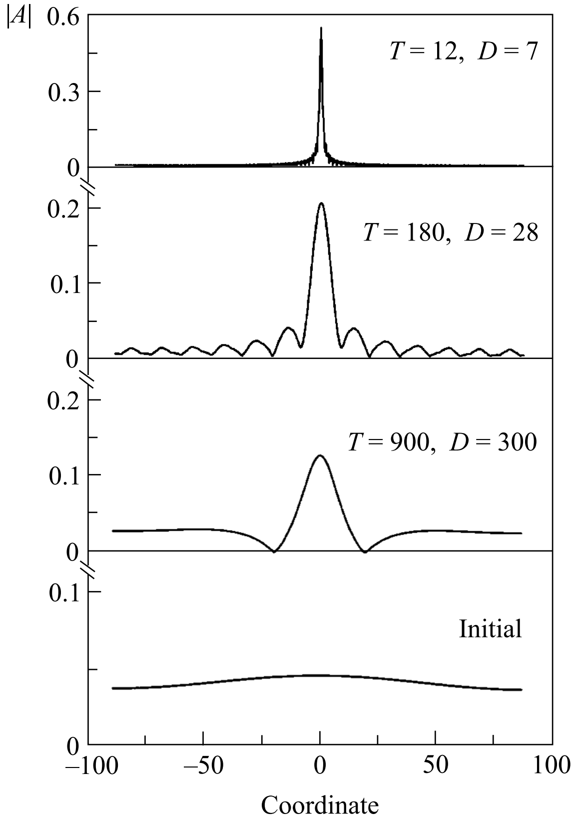

Fig.1. Development of the modulation instability for different phase indexes D. Time of appearance of the wave of maximal amplitude is provided

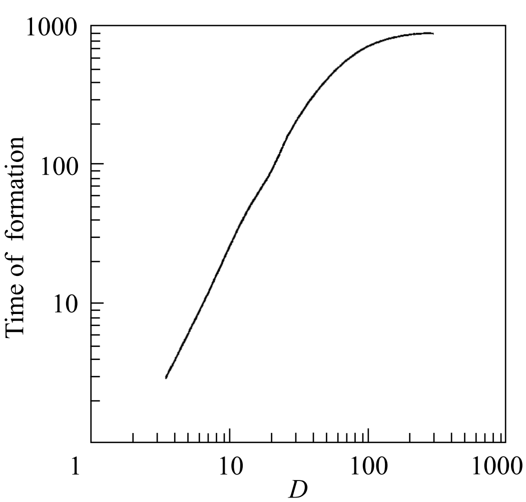

Fig.2. Time of appearance of the freak wave as a function of D

dimensionless length 176; the number of points is varied from 128 to 1024. The initial condition is

$$A(x,0) = A_0\,(1 + 0.1\cos(x/d))\exp(ix^2/D^2), \qquad (8)$$

where $A_0 = 0.043$, $d = 28$, and D is varied in wide ranges. Fig.1 shows the various forms of the freak wave for different phase indexes D. The maximal amplitude is reached at different moments of time; they are given in Fig.1. The phase modulation of the initial envelope leads to the increase of the wave amplitude and to the decrease of time of the freak wave formation (the latter is provided in Fig.2). For small D the formation time is described by the power asymptotic ($T \sim D^2/4$), as it can be shown in the linear theory; for large D it tends to the constant value defined by the modulation instability (approximately 3 T_{BF}). As it is predicted, the phase modulation of the preliminary amplitude modulated envelope leads to the formation of a more energetic wave impulse at a shorter time.

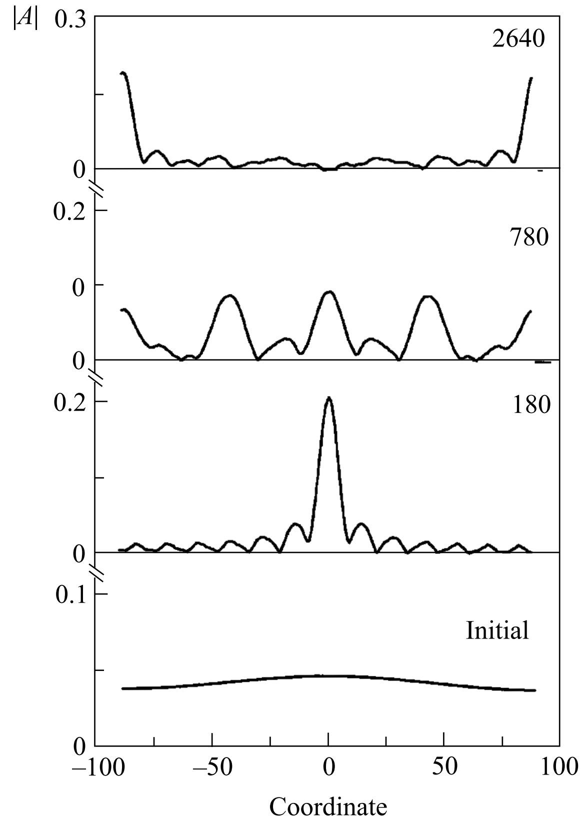

Fig.3. Wave evolution from the initial modulated disturbance ($D = 28$) at different times

For $D = 28$, the long-term nonlinear dynamics of the wave field is displayed in Fig.3. The phase modulation leads to the complex picture of the wave envelope with one or several peaks and holes, they can be considered as a group of freak waves. The time evolution of the maximal value of wave amplitude is shown in Fig.4a. The very large amplitude peaks appear several times during 12000 time units and their amplitudes decrease with time. Then the process becomes more stationary and peaks with amplitudes about 0.12 appear regularly. Fig.4b shows the time evolution within the framework of the linear theory. In this case the process of the gen-

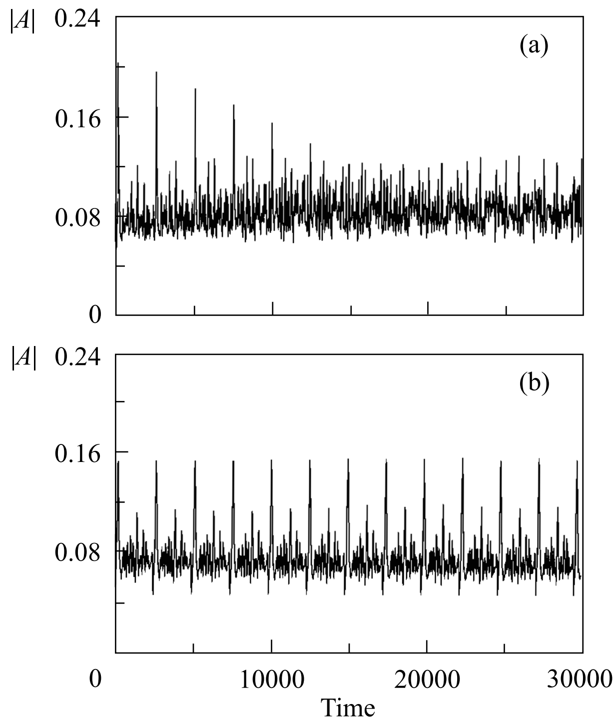

Fig.4. Time variation of the maximal amplitude in the wave packets for $D = 28$, (a) nonlinear and (b) linear cases

eration of significant peaks is almost periodic. Similar behavior is found for wide ranges of the variation of the phase index D. Physically, the role of nonlinearity in the wave field behavior can be explained as follows. The quadratic phase modulation in (8) corresponds to a linear variation of the wave number (wave frequency) with distance. On the first stage the slope of $K(x)$ increases and tends to infinity leading to the focusing (easy to show for linear waves within the framework of the kinematic equation for the wave number). After focusing, the slope of $K(x)$ changes its sign and decreases. Then, the jump in wave number is formed and the function $K(x)$ becomes multi-valued with many jumps. As a result, the wave packet in the periodic problem can focus many times, as it is shown in Fig.4b. The nonlinear effects leads to the smoothing and uniformity of the phase distribution. Here, the role of the classical modulation instability is more significant and the wave transformation is similar to the one studied in [6, 12, 13]. At this stage the amplitude of the freak wave is less than three times the amplitude of the unperturbed value.

Therefore, the effect of the phase modulation of the initial wave disturbance leads to a significant intensification of the process of the freak wave generation. The phase modulation of the wind wave field can be due to specific meteorological conditions and the relation between the observed freak waves and heavy weather conditions is very often mentioned in literature. Due to the short time of existence of the freak wave, the random forcing from the wind (this process should be studied within the framework of the forced version of the nonlinear Schrödinger equation) cannot modify radically the process of the freak wave formation from the frequency modulated disturbances at least on the first stage, meanwhile as it was shown in [27], the "usual" modulation instability is reduced in the random fields.

The one-dimensional model used cannot predict the behaviour of the wave field in two-dimensional case. The transversal instability (see [28]) may have an influence on the process of formation of a freak wave. This should become a topic for following investigations.

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