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The period map for pulse propagation in nonlinear optical DM fibres

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We have derived a simple recursion formula for the amplitude and chirp of the optical pulse propagating over a Dispersion Managed fibre with zero mean dispersion. We neglect dissipation and assume the dispersion to be constant along the adjacent legs of the waveguide, thus providing the applicability of the integrable nonlinear Schrödinger models within each leg. Choosing the legs to be long enough to ensure the formation of a self-similar profile we apply the well-known asymptotic formulas for the non-soliton initial pulses. Matching them through the interfaces of the legs we get recursion formulas for the pulse amplitude and chirp. Our analytical results are well justified by numerical simulations.

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We consider the problem of propagation of nonlinear electromagnetic pulses in optical fibres with periodically alternating dispersion (so called Dispersion Managed, or DM, fibres) and absence of dissipation. If the mean dispersion over the period vanishes and the amplitude of the pulse is small, then, according the linear theory, the complete compensation of the chromatic spreading would be achieved. In this case the map over the period of the DM fibre would be the identical transformation and the pulses would completely restore their profiles. However, to provide a low-error transmission and increase the signal-noise ratio one should use optical pulses of relatively high amplitude, thus the nonlinear effects become important and have to be accounted in the theoretical description.

The solution of the above problem is very challenging theoretically and important for many applications including industrial (see for example [1]). Many interesting results in this direction have already been obtained numerically. Maybe the most advanced analytical results were established on the basis of the Gabitov-Turitsyn approximation [2 – 4].

Our approach to the problem is entirely different. We use the fact that the corresponding nonlinear Schrödinger equations (NLS) with positive and negative dispersion are completely integrable systems. The integration scheme (the Inverse Scattering Transformation, IST, see [5]) first developed by V. E. Zakharov and A. B. Shabat in 1971 provides an adequate tool to monitor in details the pulse dynamics. Also we assume that the length z_0 of each leg of the fibre is long enough: that enable us to use the asymptotic methods for the solution of the direct and inverse scattering problems (c.f. [6]). Our goal is to find an explicit map of the pulse over the period of the DM fibre. Suppose we have a pulse profile $\hat{u}_n(t)$ at the entrance of the n-th leg (for example with negative dispersion) and we want to find the profile $\hat{u}_{n+1}(t)$of the pulse at the exit of the consequent leg with positive dispersion. We propose the following scheme:

a) Using the Lax operator L^- corresponding to the NLS equation with negative dispersion with the potential $u_n^-(t,-z_0/2) = \hat{u}_n(t)$ we solve the direct problem to determine the scattering data.

b) The scattering data have simple evolution law, thus it is easy to find the corresponding data at the exit of the n-th leg with negative dispersion.

c) Solving the inverse scattering problem we find the profile $u_n^-(t, z_0/2)$ at the exit of the n-th leg, which serves as initial data for the next leg (with positive dispersion).

d) Using the Lax operator L^+ corresponding to the NLS equation with positive dispersion with the potential $u_n^+(t,-z_0/2) = u_n^-(t, z_0/2)$ we solve the direct problem to determine the scattering data.

e) Find the corresponding scattering data at the exit of the leg with positive dispersion.

f) Solve the inverse problem in order to find the emerging pulse $\hat{u}_{n+1}(t) = u_n^+(t, z_0/2)$ after the period of the DM fibre.

We would like to emphasize that all the above mentioned steps can be explicitly performed and we also can totally control the accuracy of asymptotic solutions to the direct and inverse scattering problems. As the result we obtain an explicit recurrent relation which enable us to determine the propagation of a pulse over many periods of the DM fibre.

We have compared our analytical solution with the direct computer simulation of the problem and found a good agreement of the results. Moreover, it immediately follows from our solution that the nonlinear contribution to the chirp of the pulse accumulates with distance. This observation is in excellent agreement with numerical simulations as well.

1. Basic facts and notations. The basic model for describing of optical fibres is the nonlinear Schrödinger equation, which can be written in the dimensionless form

$$iu_z \pm u_{tt} + 2|u|^2 u = 0, \quad u = u^{\pm}(t, z), \tag{1}$$

after proper rescaling. The signs "plus" and "minus" stand for positive and negative dispersion legs respectively. The NLS equation is known to be integrable, the corresponding Lax operators

$$L^{\pm} = \begin{pmatrix} \partial_t + \frac{i\lambda}{2} & \mp i\bar{u} \\ -iu & \partial_t - \frac{i\lambda}{2} \end{pmatrix}, \quad L^{\pm}\Psi = 0, \tag{2}$$

for the spectral transform were found by V.E.Zakharov and A.B.Shabat about thirty years ago. We shall consider pulses rapidly decaying as $|t| \to \infty$ and assume that the spectral problem (2) does not have discrete eigen-values. That is the most interesting region of pulse parameters which are commonly used in optical systems of telecommunications [1]. The continuous spectral data yields the standard z-evolution [5]

$$\Psi \to \begin{pmatrix} 0 \\ \exp\left(\frac{i\lambda t}{2}\right) \end{pmatrix}, \quad t \to +\infty,$$

$$\Psi \to \begin{pmatrix} b^{\pm}(\lambda) e^{\pm i\lambda^2 z} \exp\left(-\frac{i\lambda t}{2}\right) \\ a^{\pm}(\lambda) \exp\left(\frac{i\lambda t}{2}\right) \end{pmatrix}, \quad t \to -\infty. \tag{3}$$

and satisfy the unitary condition $|a^{\pm}|^2 \pm |b^{\pm}|^2 = 1$ together with analytically of the $a^{\pm}$ functions at the upper-half plane $\operatorname{Im} \lambda > 0$.

Here we shall assume that both legs with positive and negative dispersion have equal length z_0 and $\delta = z_0^{-1}$ is a small dimensionless parameter of our theory $\delta \ll 1$. It is convenient to introduce a notation for the small parameter $\varepsilon = z_0^{-1} \ln z_0$.

Suppose we have a pulse $u(t, 0)$ given at the middle of a leg with positive dispersion. We assume that this initial data does not contain solitons. Then at the exit of the leg, i.e. at $z = z_0/2$ it evaluates to the following asymptotic form (see [7 − 10])

$$u(t, \frac{z_0}{2}) = \delta^{1/2} \left[\alpha(\xi) - \varepsilon \frac{(\alpha^4(\xi))_{\xi\xi}}{4\alpha(\xi)} + O(\delta) \right] \times$$

$$\times \exp i \left\{ \frac{\xi^2}{2\delta} + \varepsilon \frac{2\alpha^2(\xi)}{2\delta} + O(1) \right\}, \quad \xi = t\delta, \tag{4}$$

where the amplitude $\alpha(\xi)$ can be expressed in terms of scattering data $\alpha^2(\xi) = -\frac{1}{\pi} \ln |a^{+}(\xi)|$ for the corresponding spectral problem.

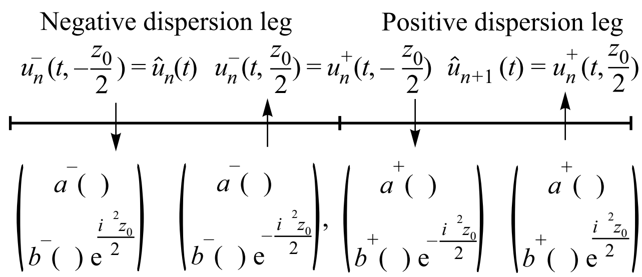

Fig.1

Solution (4) suggests the following forms for functions $u_n^-(t, -z_0/2), u_n^-(t, z_0/2) = u_n^+(t, -z_0/2)$ and $u_n^+(t, z_0/2)$ (see Fig.1):

$$u_n^-(t, -z_0/2) =$$

$$= \delta^{1/2} (A_n^-(\xi, \varepsilon) + O(\delta)) \exp \left\{ \frac{iS_n^-(\xi, \varepsilon)}{2\delta} \right\}, \tag{5}$$

$$u_n^-(t, z_0/2) = u_n^+(t, -z_0/2) =$$

$$= \delta^{1/2} (A_n(\xi, \varepsilon) + O(\delta)) \exp \left\{ \frac{iS_n(\xi, \varepsilon)}{2\delta} \right\}, \tag{6}$$

$$u_n^+(t, z_0/2) =$$

$$= \delta^{1/2} (A_n^+(\xi, \varepsilon) + O(\delta)) \exp \left\{ \frac{iS_n^+(\xi, \varepsilon)}{2\delta} \right\}, \tag{7}$$

where

$$S_n^{\pm}(\xi, \varepsilon) = \xi^2 + 2\varepsilon\gamma_n^{\pm}(\xi) + \delta\varphi_n^{\pm} + O(\varepsilon^2),$$

$$S_n(\xi, \varepsilon) = \xi^2 + 2\varepsilon\gamma_n(\xi) + \delta\varphi_n + O(\varepsilon^2).$$

Functions $\gamma_n(\xi), \gamma_n^{\pm}(\xi)$ we call the nonlinear logarithmic phases and $A_n(\xi, \varepsilon), A_n^{\pm}(\xi, \varepsilon)$ the amplitudes. The representations (5)–(7) can be rigorously justified by the asymptotic solutions of the corresponding inverse scattering problems providing the conditions

$$\varepsilon|\frac{d^2\gamma_n^{\pm}}{d\xi^2}| < 1\,, \qquad \varepsilon|\frac{d^2\gamma_n}{d\xi^2}| < 1. \tag{8}$$

Conditions (8) we call the small nonlinear chirp conditions. The origin of these conditions is the following. We apply the WKB method to solve the corresponding direct and inverse scattering problem and conditions (8) guarantee that only one turning point contributes in the result.

2. In sections a)–f) which correspond to the steps described in Fig. 1 we work with one period of the DM fibre and omit the index n in our notations. We will restore the index in the section devoted to the recursion relation.

a. In order to solve the direct scattering problem corresponding to potential $u_n^-(t,-z_0/2)$ we substitute (5) in the spectral problem (2), (3) and perform the change of variables. As the result we get

$$\begin{aligned}\sqrt{\delta}\psi_1' &= -iA^- \exp\left\{\frac{-iS^-(\xi,\varepsilon)+2i\lambda\xi}{2\delta}\right\}\psi_2,\\ \sqrt{\delta}\psi_2' &= iA^- \exp\left\{\frac{iS^-(\xi,\varepsilon)-2i\lambda\xi}{2\delta}\right\}\psi_1,\end{aligned} \tag{9}$$

where $' = d/d\xi$ and $\Psi = (\psi_1 e^{it\lambda/2}, \psi_2 e^{-it\lambda/2})^T$,

$$\begin{aligned}&\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{t\to+\infty} \to \begin{pmatrix}0\\ 1\end{pmatrix};\\ &\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{t\to-\infty} \to \begin{pmatrix}\mathrm{b}^-(\lambda,-\frac{z_0}{2})\exp\left\{\frac{i\lambda^2}{2\delta}\right\}\\ \mathrm{a}^-(\lambda,-\frac{z_0}{2})\end{pmatrix}.\end{aligned} \tag{10}$$

Following the pioneer work of V.E.Zakharov and S.V.Manakov we solve the system (9) by the WKB-method. The stationary (or turning) points here are found from the equation $S_+' - 2\lambda = 0$, which is

$$\xi + \varepsilon\gamma_+'(\xi) - \lambda = 0. \tag{11}$$

Condition (8) guarantee that there is only one turning point

$$\begin{aligned}&\mathrm{a}^-(\lambda,-\frac{z_0}{2}) =\\ &= \exp\left\{-i\int_{-\infty}^{+\infty}\frac{A_+^2(p,\varepsilon)}{p-\lambda+\varepsilon\gamma_+'(p)}dp\right\},\ \mathrm{Im}\,\lambda > 0,\end{aligned} \tag{12}$$

$$\begin{aligned}&\mathrm{b}^-(\lambda,-\frac{z_0}{2}) =\\ &=\beta^-(\lambda)\exp\left(\frac{-2i\varepsilon A_+^2(\lambda,\varepsilon)-2i\varepsilon\gamma_+(\lambda)}{2\delta}\right),\ \mathrm{Im}\,\lambda = 0.\end{aligned} \tag{13}$$

where the function $\beta^-(\lambda)$ can be found explicitly, but we do not use it in the following consideration.

c. Using the asymptotic solution of the inverse scattering problem for operator L^- (actually following the original method prosed in [6]) we can relate the amplitude $A(\xi,\varepsilon)$ and the logarithmic phase $\gamma(\xi)$ at the exit of the leg with negative dispersion with $A^-(\xi,\varepsilon)$ and $\gamma^-(\xi)$

$$(A^-(\xi,0))^2 + (A(-\xi,0))^2 + \gamma^-(\xi) - \gamma(-\xi) = 0, \tag{14}$$

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{(A(p,\varepsilon))^2}{p+\lambda+i0-\varepsilon(\gamma(p))'}dp =\\ &=\int_{-\infty}^{+\infty}\frac{(A^-(p,\varepsilon))^2}{p-\lambda-i0+\varepsilon(\gamma^-(p))'}dp,\end{aligned} \tag{15}$$

where $\mathrm{Im}\,\lambda = 0$.

Under the assumption of the *small chirp* (8) it is easy to solve integral equation (15). Since each functional equation

$$\xi = \mu + \varepsilon(\gamma^-(\mu))', \quad \xi = \nu + \varepsilon(\gamma(-\nu))',$$

has only one root, the real parts of the singular integrals (15) yield

$$(A(\nu,\varepsilon))^2 = \frac{1-\varepsilon(\gamma(\nu))''}{1+\varepsilon(\gamma^-(\mu))''}(A^-(\mu,\varepsilon))^2. \tag{16}$$

Eq.(14),(16) can be easily solved in order to express $A(\xi,\varepsilon)$ and $\gamma(\xi)$ in terms of $A^-(\xi,\varepsilon)$ and $\gamma^-(\xi)$.

d-f. Very similar to the sections **a-c** we solve the direct and inverse scattering problems for operator L^+ on the leg with positive dispersion. It enables us to relate the amplitude $A(\xi,\varepsilon)$ and logarithmic phase $\gamma(\xi)$ given at the entrance of the leg with the exit values $A^+(\xi,\varepsilon)$ and $\gamma^+(\xi)$. The result is very similar to (14), (15):

$$(A^+(\xi,0))^2 + (A(-\xi,0))^2 + \gamma(\xi) - \gamma^+(\xi) = 0, \tag{17}$$

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{(A^+(p,\varepsilon))^2}{p-\lambda-i0+\varepsilon(\gamma^+(p))'}dp =\\ &\int_{-\infty}^{+\infty}\frac{(A(p,\varepsilon))^2}{p+\lambda+i0-\varepsilon(\gamma(p))'}dp,\end{aligned} \tag{18}$$

where $\operatorname{Im}\lambda = 0$.

Providing the small chirp condition (8) equation (18) can be easily solved

$$(A^{+}(\mu,\varepsilon))^2 = \frac{1+\varepsilon(\gamma^{+}(\mu))''}{1-\varepsilon(\gamma(\nu))''}(A(\nu,\varepsilon))^2. \qquad (19)$$

where μ and ν are unique solutions of equations

$$\xi = \mu + \varepsilon(\gamma^{+}(\mu))', \quad \xi = \nu + \varepsilon(\gamma(-\nu))'.$$

3. Assembling the results obtained in the previous section and restoring the index n which enumerates the sections of the DM fibre we arrive to the recursion relations for the amplitudes and logarithmic phases of the pulse

$$(A_n^{-}(\xi,0))^2 + (A_n(-\xi,0))^2 + \gamma_n^{-}(\xi) - \gamma_n(-\xi) = 0,$$
$$(A_n^{+}(\xi,0))^2 + (A_n(-\xi,0))^2 + \gamma_n(-\xi) - \gamma_n^{+}(\xi) = 0,$$
$$(A_n^{-}(\nu_n,\varepsilon))^2 = \frac{1-\varepsilon(\gamma_n^{-}(\nu_n))''}{1+\varepsilon(\gamma_n(\hat\mu_n))''}(A_n(\hat\mu_n,\varepsilon))^2,$$
$$(A_n^{+}(\mu_n,\varepsilon))^2 = \frac{1+\varepsilon(\gamma_n^{+}(\mu_n))''}{1-\varepsilon(\gamma_n(\hat\nu_n))''}(A_n(\hat\nu_n,\varepsilon))^2,$$
$$A_{n+1}^{-}(\xi,\varepsilon) = A_n^{+}(\xi,\varepsilon),$$
$$\gamma_{n+1}^{-}(\xi) = \gamma_n^{+}(\xi).$$

Here $\nu_n, \hat\mu_n, \mu_n, \hat\nu_n$ are solutions of the following equations

$$\xi = \hat\mu_n + \varepsilon(\gamma_n(\hat\mu_n))', \quad \xi = \nu_n + \varepsilon(\gamma_n^{-}(-\nu_n))',$$
$$\xi = \mu_n + \varepsilon(\gamma_n^{+}(\mu_n))', \quad \xi = \hat\nu_n + \varepsilon(\gamma_n(-\hat\nu_n))'.$$

These solutions are unique providing the small chirp conditions (8).

It follows from the above recursion relation that the small chirp condition is not uniformly valid and will brake down at some distance, even if it satisfies well for the input pulse. Indeed, suppose the initial profile is of the form (4)

$$\gamma_1^{-}(\xi) = \alpha^2(\xi), \quad A_1^{-}(\xi,\varepsilon) = \alpha(\xi) - \varepsilon\frac{(\alpha^4(\xi))_{\xi\xi}}{4\alpha(\xi)}$$

and satisfies the condition (8). It follows from the above recursion relations that after n periods of propagation we receive

$$\gamma_{n+1}(\xi) = (2n+1)\alpha^2(\xi),$$

$$A_{n+1}(\xi,\varepsilon) = \alpha(\xi) - \varepsilon\frac{(2n+1)(\alpha^4(\xi))_{\xi\xi}}{4\alpha(\xi)} + O(\varepsilon^2),$$

$$\gamma_{n+1}^{+}(\xi) = (2n+3)\alpha^2(\xi),$$

$$A_{n+1}^{+}(\xi,\varepsilon) = \alpha(\xi) + \varepsilon\frac{(2n+3)(\alpha^4(\xi))_{\xi\xi}}{4\alpha(\xi)} + O(\varepsilon^2).$$

It is clear that for $n \sim \max((\gamma^{-}(\xi))'')\varepsilon^{-1}$ the condition (8) fails and we have to account other turning points in the WKB analysis, which is beyond of the scope of this paper. According our theory, the nonlinear contribution to the chirp $(\gamma''(\xi))$ is increasing with the propagation of the pulse along the DM fibre.

4. Direct numerical simulations of the NLS equation with alternating dispersion are in a good agreement with our analytical results. The standard split-step pseudo-spectral scheme were used for the numerical solution of the problem. The initial condition was taken in the form

$$u(t,0) = 0.42\exp\left(-\frac{t^2}{16} + \frac{it}{4}\right). \qquad (20)$$

By the numerical solution of the direct scattering problem we have checked that the spectral problem with potential (20) does not have discrete eigen-values. The length of each leg was taken large enough $z_0 = 400$, so that

$$\delta = \frac{1}{400} = 0.0025, \quad \varepsilon = \frac{\ln 400}{400} \approx 0.015.$$

We compared the numerical profiles of the pulse at the points of the interfaces with the analytical expressions obtained via the recurrence relations. The results are shown on Fig.2. The increasing of the nonlinear contribution to the chirp obtained analytically is in a perfect agreement with our numerical simulations.

5. Conclusion. The study of the DM maps is a new approach to the problem of description of pulse propagation in optical fibres with dispersion management. Our paper is a first and quite modest step in this direction, but we believe that it can be useful for further understanding and even description of the optical pulse evolution over a number of initial DM–periods of its propagation in DM systems. It is very important that we can rigorously control the accuracy (more precisely, the residue terms) in our asymptotic formulas. The small nonlinear chirp condition (8) imposes natural limitations on the applicability of our theory. We are still unable to control analytically the formation of true DM–solitons. There are significant complications in the application of the WKB asymptotic technique while dealing with multiple stationary points which contribute in the solutions of the corresponding direct and inverse scattering problems. The work in this direction is currently in progress. We hope to overcome this drawback and receive a functional discrete dynamical system for the corresponding amplitudes and phases, which would have the DM-soliton as its attractor. That would also open a perspective to solve a challenging problem of the description of the DM–solitons interaction.

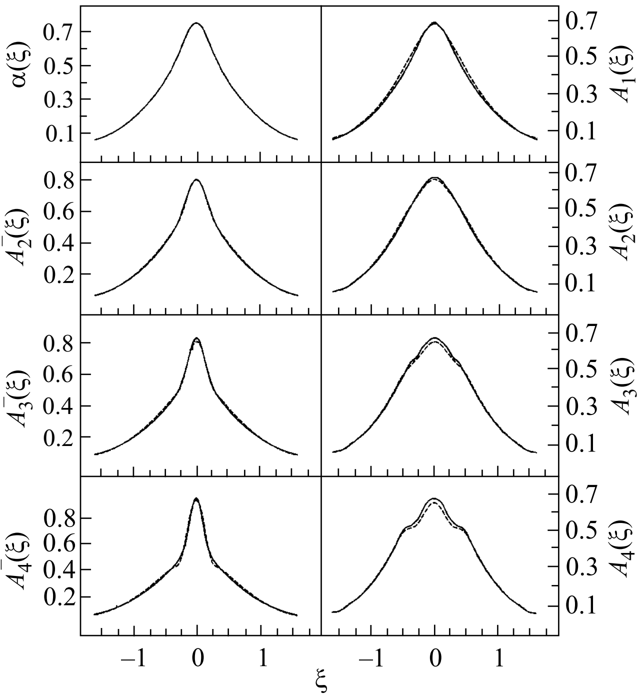

Fig.2. The subsequent pulse amplitude profiles. The solid line corresponds to numeric simulation, dashed line – to asymptotic expansions by the recurrence formulas

Another promising direction of the development is the study of systems with amplifiers and filters at the junction points of the legs with different dispersion. It is easy to control the pulse dynamics by multiplying the transmission coefficient $a(\lambda)$ on a fixed constant (the amplification factor), or even on a function (filtering). As well one can made obvious modifications of the above theory to describe DM systems with any constant-wise profile of dispersion.

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