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Robustness of the inflationary perturbation spectrum to trans-Planckian physics

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It is investigated if predictions of the inflationary scenario regarding spectra of scalar and tensor perturbations generated from quantum vacuum fluctuations are robust with respect to a modification of the dispersion law for frequencies beyond the Planck scale. For a large class of such modifications of special and general relativity, for which the WKB condition is not violated at super-high frequencies, the predictions remain unchanged. The opposite possibility is excluded by the absence of large amount of created particles due to the present Universe expansion. Creation of particles in the quantum state minimizing the energy density of a given mode at the moment of Planck boundary crossing is prohibited by the latter argument, too (contrary to creation in the adiabatic vacuum state which is very small now).

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Approximately flat spectrum of scalar and tensor perturbations generated from quantum vacuum fluctuations at a quasi-de Sitter (inflationary) state in the early Universe is certainly the most important prediction of the inflationary scenario since it can be directly tested and falsified using observational data. Fortunately, all existing and constantly accumulating data, instead of falsifying, confirm these predictions (within observational errors). The other observational prediction of the simplest variants of the inflationary scenario – the approximate flatness of the Universe, $|\Omega_{tot} - 1| \ll 1$ – is actually a consequence of the first one since an isotropic part of the spatial curvature may be considered as a monopole perturbation with respect to the spatially flat Friedmann–Robertson–Walker (FRW) background. Note that the first quantitatively correct derivation of perturbation spectra *after* inflation was first obtained in [1] in the case of tensor perturbations (gravitational waves) and in [2] in the case of scalar (adiabatic) perturbations. For completeness, one should mention two important intermediate steps on the way to the right answer for scalar perturbations made between 1979 and 1982: in [3], the first estimate of scalar perturbations after inflation was made according to which scalar and tensor perturbations are of the same order of magnitude, while in [4], the spectrum of scalar perturbations *during* inflation was calculated for the Starobinsky inflationary model [5] (however, the actual amplitude of scalar perturbations *after* inflation was still significantly underestimated in both these papers).

Therefore, it is very important to investigate the validity of assumptions on which this prediction is based[2)]. All derivations of perturbation spectra use quantum field theory in classical curved space-time or semiclassical quantum cosmology. Both these approaches are valid and essentially equivalent if $H \ll M_P$ where $H \equiv \dot{a}/a$, $a(t)$ is the scale factor of a flat FRW cosmological model, the dot denotes time derivative, $M_P = \sqrt{G}$, and $\hbar = c = 1$ is put throughout the paper. On the other hand, comparison of the predicted spectrum with observational data shows that H should be less than $\sim 10^{-5} M_P$ at least during last 70 e-folds of inflation. So, the assumption $H \ll M_P$ is *required* and *self-consistent*, if we are speaking about inflationary models having relation to reality. Recently it was questioned if inflationary predictions are robust with respect to a change in the so-called "trans-Planckian physics". What is meant by this term is some *ad hoc* modification of special and general relativity leading to violation of the Lorentz invariance and to deviation of the dispersion law $\omega(k)$ for field quanta from the linear one for frequencies (energies) $\omega > M_P$, where k is a particle wave number (momentum). In the absence of the Lorentz invariance, a preferred system of reference appears (in which this dispersion law is written). Usually, it is identified with the basic cosmological system of reference which is at rest with respect to spatially averaged matter in the Universe.

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[2)]Results of the present paper partially intersect with those obtained in the recent papers [6, 7] (which appeared when this paper was prepared for publication) and are in general agreement with them whenever intersect.

Initially, "trans-Planckian physics" was introduced to obtain a new way of derivation of the Hawking radiation from black holes. In this case, it was shown that the spectrum of the Hawking radiation does not depend on a concrete form of the dispersion law $\omega(k)$ at $k \to \infty$ [8–10]. On the other hand, an opposite result was recently claimed in [11] regarding the inflationary perturbation spectrum. There exists no self-consistent theory of such a modification leading to some unique dispersion law $\omega(k)$, but arguments showing that this possibility should not be considered as logically impossible are based either on higher-dimensional models of the Universe (see, e.g., the recent paper [12]), or on condensed matter analogs of gravity [13, 14] which do not have too much symmetry at the most fundamental level. So, $\omega(k)$ should be considered as some fixed but unknown function at the present state-of-the-art.

The very possibility of trans-Planckian physics affecting the (supposedly known) sub-Planckian one is due to the expansion of the Universe. This expansion gradually shifts all modes of quantum fields from the former region to the latter one. Really, for a FRW model with the metric

$$ds^2 = dt^2 - a^2(t)\, dl^2 \tag{1}$$

where dl^2 is the 3D Euclidean space interval (spatial curvature may be always neglected), spatial dependence of a given mode of a quantum field may be taken as $\exp(in_\mu x^\mu)$, $\mu = 1,2,3$. Then the frequency $\omega = n/a(t)$, $n = |\mathbf{n}| = \mathrm{const}$ in the ultra-relativistic (but still Lorentzian) limit. This red-shifting occurs equally well in the early and the present-day Universe. So, any effect connected with trans-Planckian physics can be observed now, too; inflation (i.e., the epoch when $|\dot{H}| \ll H^2$) is not specific for that at all.

We will model metric fluctuations by a massless, minimally coupled scalar field satisfying the equation $\nabla_i \nabla^i \phi = 0$. This form is sufficient for both scalar perturbations for which the effective mass satisfies the condition $|m^2| \ll H^2$ necessary for inflation, and for tensor perturbations since their amplitude satisfies the same wave equation in the FRW Universe filled by any matter with no non-diagonal pressure perturbations ($\delta p_{\mu\nu} \propto \delta_{\mu\nu}$). Also it is assumed that $H \ll M_P$. Then the equation for the time-dependent part of ϕ_n reads

$$\ddot{\phi}_n + 3H\dot{\phi}_n + \omega^2\left(\frac{n}{a}\right)\phi_n = 0 \tag{2}$$

with $\omega(k) = k$ for $\omega \ll M_P$. Solutions of this equation have the WKB form for $H \ll \omega \ll M_P$:

$$\phi_n = \frac{\alpha_n}{\sqrt{2n}a} e^{-in\eta} + \frac{\beta_n}{\sqrt{2n}a} e^{in\eta}, \quad \eta = \int \frac{dt}{a(t)}, \tag{3}$$

where α_n, $\beta_n = \mathrm{const}$ and $|\alpha_n|^2 - |\beta_n|^2 = 1$ for any quantum state, if the quantum field $\hat{\phi}$ is second quantized and $\phi_n \exp(in_\mu x^\mu)(2\pi)^{-3/2}$ is the c-number coefficient of the Fock annihilation operator $\hat{a}_\mathbf{n}$. The average number of created pairs is $N(n) = |\beta_n|^2$. Therefore, whatever the trans-Planckian physics is (namely, what is the form of $\omega(k)$ and what is the initial condition for ϕ_n for $t \to -\infty$), once $\omega \ll M_P$, we may say that the field mode (3) emerges from the Planck boundary $n = M_P a$ in some quantum state characterized by α_n and β_n. In particular, the rate of growth of the average energy density of particles with $\omega \ll M_P$ is

$$\frac{d(<\varepsilon> a^4)}{a^4\, dt} = \frac{gM^4H}{2\pi^2} N(n)|_{n=Ma}, \tag{4}$$

where $g = 1$ for scalars and $g = 2$ for gravitons. M is an auxiliary mass satisfying $H \ll M < M_P$ for which $\omega(M) = M$ with sufficient accuracy (for estimates, we will take $M = M_P$). It follows from time translation invariance that $N^{(0)}(n)$ is independent on n. Here $N^{(0)}$ means the part of $N(n)$ which does not depend on background space-time curvature at the moment of Planck boundary crossing ($n = M_P a$).

Let us first consider the case when the WKB condition for ϕ_n is satisfied for all $n \gg Ha$ including the trans-Planckian region $n > M_P a$. Then the natural and self-consistent choice of the initial condition for ϕ_n is the adiabatic vacuum for $t \to -\infty$:

$$\phi_n = \frac{1}{\sqrt{2\omega_n a^3}} \exp(-i\int \omega_n dt). \tag{5}$$

Note that this mode is not in the minimum energy density state at finite t, in particular, at the Planck boundary crossing (we return to the discussion of this point below). Eq. (5) reduces to Eq. (3) with $\beta_n = 0$, $\alpha_n = 1$ in the sub-Planckian region. Then it just coincides with the initial condition for ϕ_n used in the standard calculation of the spectrum of inflationary perturbations. Thus, no correction to the standard result arises in this case irrespective of the form of $\omega(n/a)$.

The necessary condition for the WKB behaviour is $|\dot{\omega}| \ll \omega^2$, or

$$\frac{H|d(1/\omega(k))|}{d\ln k} \ll 1, \quad k = n/a \tag{6}$$

for all $k \gg M_P$. Since H/M_P is already a small parameter and $\omega(k)$ presumably does not depend on H for $k \gg H$, this inequality is satisfied practically always, if ω does not become zero either for $k \to \infty$ or at some finite $k_0 > M_P$ (another dangerous case is when $d\omega/dk$ diverges at a finite $k = k_0$, in particular, if $\omega \propto (k_0 - k)^\gamma$ with $-1 < \gamma < 0$ or $\omega \approx \omega_0 + \omega_1(k_0 - k)^\gamma$, $0 < \gamma < 1$).

As a consequence, $N^{(0)} = 0$ for the dispersion law $\omega(k) = M \tanh^{1/m}[(k/M)^m]$, $m > 0$ proposed by Unruh [8], for $\omega^2 = k^2[1 + b_m(k/M)^{2m}]$ with positive m and b_m considered in [10, 11], and for the dependence $\omega^2 = [k \ln(1 + k/M)]^2$ introduced in [15].

Still, there exist exceptional forms of $\omega(k)$ for which the WKB behaviour is not valid for some $k > M_P$. In particular, this refers to the case $\omega^2 = k^2[1 + b_m(k/M)^{2m}]$ with $b_m < 0$ and to the dispersion law introduced in the recent paper [16] for which $\omega(k) \to 0$ at $k \to \infty$. *A priori*, such a possibility may not be excluded. Then there is *no* preferred initial condition for ϕ_n, and it is not possible to define a unique initial vacuum state. So, in this case $N^{(0)} \neq 0$ generically, i.e., creation of pairs in the expanding Universe occurs due to trans-Planckian physics.

However, nature tells us that such an effect is infinitesimally small, if exists at all. Really, from the evident condition that created ultra-relativistic particles do not significantly contribute to the present energy density in the Universe, it follows that $N^{(0)} \lesssim H_0^2/M_P^2 \sim 10^{-122}$ where $H_0 = H(t = t_0)$ is the Hubble constant. Thus, curvature independent particle creation in the expanding Universe due to trans-Planckian physics is very strongly suppressed in any case because of observational data. Of course, the corresponding change in the inflationary perturbation spectrum is negligible, too (the relative correction is $\sim |\beta_n| = \sqrt{N^{(0)}}$).

Finally, let us consider a more subtle effect: creation of particles due to both trans-Planckian physics and background space-time curvature in the expanding Universe. Then $N(n) \sim H^2/M_P^2$ where H is estimated at the moment of high energy boundary crossing $n = Ma(t)$. Certainly, corrections to the inflationary spectrum are already negligible ($\sim H/M_P < 10^{-5}$) in this case. Nevertheless, even such a small effect can be significantly restricted. An example of this effect arises if we would assume that modes crossing the boundary $n = Ma$ are in the exactly minimum energy density state just at this moment, i.e., $\dot{\phi}_n = -in\phi_n/a = -i\sqrt{n/2}\, a^{-2}$ and $\varepsilon_n \equiv (|\dot{\phi}_n|^2 + n^2a^{-2}|\phi_n|^2)/2 = n/2a^4$ for each mode at the moment $t = t_n$ when $n = Ma$. On the other hand, the adiabatic vacuum for each mode has the larger energy density

$$\varepsilon_n = \frac{n}{2a^4}\left(1 + \frac{H^2a^2}{2n^2}\right) \tag{7}$$

(see, e.g., [17, 18]). Note that this excess is due to vacuum polarization only. Of course, this assumption may be immediately criticized from the logical point of view since such a state ceases to diagonalize the mode Hamiltonian and minimize its energy density for all other moments of time $t \neq t_n$. Nevertheless, let us consider its implications.

Writing as, e.g., in [17]:

$$\phi_n(t) = (2\omega_n a^3)^{-1/2} \times$$
$$\times \left(\alpha_n(t) \exp(-i \textstyle\int \omega_n\, dt) + \beta_n(t) \exp(i \textstyle\int \omega_n\, dt)\right), \tag{8}$$

$$\dot{\phi}_n(t) = -i \left(\frac{\omega_n}{2a^3}\right)^{1/2} \times$$
$$\times \left(\alpha_n(t) \exp(-i \textstyle\int \omega_n\, dt) - \beta_n(t) \exp(i \textstyle\int \omega_n\, dt)\right), \tag{9}$$

so that $\alpha_n(t_n) = 1$, $\beta_n(t_n) = 0$ for the Heisenberg quantum state of each mode $|\psi_n>$ which minimizes its Hamiltonian and energy density at the moment $t = t_n$ when $n = Ma$, we obtain the following system of equations for $\alpha_n(t)$ and $\beta_n(t)$:

$$\dot{\alpha}_n = \frac{1}{2}\left(\frac{\dot{\omega}}{\omega} + 3\frac{\dot{a}}{a}\right) e^{2i\int \omega_n dt}\beta_n, \tag{10}$$

$$\dot{\beta}_n = \frac{1}{2}\left(\frac{\dot{\omega}}{\omega} + 3\frac{\dot{a}}{a}\right) e^{-2i\int \omega_n dt}\alpha_n, \tag{11}$$

with the additional condition $|\alpha_n|^2 - |\beta_n|^2 = 1$. If $\omega \gg H$, β_n is small and $\alpha_n \approx 1$. For $t \geq t_n$, one may take $\omega_n \approx n/a$. Then $\beta_n = -(iH(t_n)/2M)\exp(-2i\eta(t_n))$ plus a strongly oscillating term. So,

$$N(n) = |\beta_n(\infty)|^2 = H^2(t_n)/4M^2. \tag{12}$$

If the cosmological constant is neglected and the present law of the Universe expansion is taken as $a(t) \propto t^{2/3}$, then $N(n) \propto n^{-3}$ for particle energies close to M_P at the present time. Integrating Eq. (4) with $N(n)$ from Eq. (12), we obtain $\varepsilon_g = M^2/9\pi^2t^2$ for gravitons. For $M \sim M_P$, $\varepsilon_g \sim H^2/G$ that contradicts the assumption that $a(t) \propto t^{2/3}$. In other words, this model of particle creation by trans-Planckian physics results in a significant part of the present total energy density of matter in the Universe being contained in gravitons with energies $\sim M_P$ that is not compatible with the observed behaviour of $a(t)$. Similar arguments show that there may be no term

$$N(n) = N^{(1)}(n)\frac{|R(t_n)|}{M_P^2} \tag{13}$$

with $N^{(1)} \sim 1$ in Eq. (4). Here R is the scalar curvature.

On the other hand, for the adiabatic vacuum state in the WKB regime, the quantity $\beta_n = iH(t)a(t)\exp(-2in\eta)/2n$ in the leading order, so it approaches zero for $t \to \infty$. Note that creation of real gravitons does occur in the next order ($N = N^{(2)}(n)R^2/M_P^2$), and even without any violation of the Lorentz invariance [19]. In the latter case, the effect is due to violation of the WKB approximation at ultra-low, not ultra-high, frequencies $\omega \sim H$. Also, the notion of "vacuum" as the state of a minimum energy density may be restored in the following non-rigorous sense: the adiabatic vacuum of each mode n in the WKB regime has the lowest energy density compared to other quantum states, if the energy density is averaged ("coarse grained") over a time interval $\Delta t \gg \omega_n^{-1}$ in accordance with the energy uncertainty relation.

So, whatever occurs in the trans-Planckian region, observational evidence show that creation of particles due to mode transition from the trans-Planckian region to the sub-Planckian one is absent with a very high accuracy. Standard predictions about perturbations generated during inflation are not altered by this hypothetical mechanism, too.

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