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Mesoscopic Casimir forces in quantum vacuum

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Traditionally it is assumed that the Casimir vacuum pressure does not depend on the ultraviolet cut-off. There are, however, some arguments that the effect actually depends on the regularization procedure and thus on the trans-Planckian physics. We provide the condensed matter example where the Casimir forces do explicitly depend on the microscopic (correspondingly trans-Planckian) physics due to the mesoscopic finite-N effects, where N is the number of bare particles in condensed matter (or correspondingly the number of the elements comprising the quantum vacuum). The finite-N effects lead to mesoscopic fluctuations of the vacuum pressure. The amplitude of the mesoscopic flustuations of the Casimir force in a system with linear dimension L is by the factor $N^{1/3} \sim L/a_P$ larger than the traditional value of the Casimir force given by effective theory, where $a_P = \hbar/p_P$ is the interatomic distance which plays the role of the Planck length.

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Introduction. The attractive force between two parallel metallic plates in vacuum induced by the vacuum fluctuations of the electromagnetic field has been predicted by Casimir in 1948 [1]. The calculation of the vacuum presure is based on the regularization schemes, which allows to separate the effect of the low-energy modes of the vacuum from the huge diverging contribution of the high-energy degrees of the freedom. There are different regularization schemes: Riemann's zeta-function regularization; introduction of the exponential cutoff; dimensional regularization, etc. People are happy when different regularization schemes give the same results. But this is not always so (see e.g. [2 – 4], and in particular the divergencies occurring for spherical geometry in even spatial dimension are not cancelled [5, 6]). This raises some criticism against the regularization methods [7] or even some doubts concerning the existence and the magnitude of the Casimir effect.

The same type of the Casimir effect arises in condensed matter, due to thermal (see review paper [8]) or/and quantum fluctuations. When consindering the analog of the Casimir effect in condensed matter, the following correspondence must be taken into account. The ground state of quantum liquid corresponds to the vacuum of quantum field theory. The low-energy bosonic and fermionic quasiparticles in quantum liquid correspond to matter. The low energy modes with linear spectrum $\omega = c_s p$ can be described by the relativistic-type effective theory. The speed of sound c_s or of other collective modes (spin waves, etc.) plays the role of the speed of light. This "speed of light" is the "fundamental constant" which enters the effective theory (quantum hydrodynamics in quantum liquids, or electromagnetic theory in real vacuum). The fundamental constants of the effective theory can be in principle calculated using the microscopic physics, an analog of the transPlanckian physics. The effective theory is valid only at low energy, which is much smaller than the "Planck cutoff". In quantum liquids the analog of the Planck energy scale E_P is determined either by the mass m of the atom of the liquid, $E_P \equiv mc_s^2$, or by the Debye energy, $E_P \equiv \hbar c_s/a_P$, where a_P is the interatomic distance which plays the role of the Planck length [9].

In some cases the analogy between effective theories in quantum vacuum and in quantum liquids becomes exact. For example, the low-energy fermionic and bosonic collective modes can correspond to the chiral fermions, and gravitational and gauge fields. This allows to simulate in condensed matter such phenomena as chiral anomaly, event horizon, etc. (see review [9]).

The advantage of the quantum liquid is that the structure of the quantum vacuum is known at least in principle. That is why one can calculate everything starting from the first principle microscopic theory. For example, one can calculate the vacuum energy under different external conditions, without invoking any cut-off or regularization scheme. Then one can compare the results with what can be obtained within the effective theory which deals only with the low-energy phenomena. The latter requires the regularization scheme in order to cancel the ultraviolet divergency, and thus one can judge whether and which of regularization schemes are physically relevant.

The traditional Casimir effects deals with the low energy massless modes. The typical massless modes in quantum liquid are sound waves. The acoustic field is described by the effective theory and corresponds to the massless scalar field. The walls provide the boundary conditions for the sound wave mode, usually these are the Neumann boundary conditions. Because of the quantum hydrodynamic fluctuations there must be the Casimir force between two parallel plates immersed in the quantum liquid. Within the effective theory the Casimir force is given by the same equation as the Casimir force acting between the conducting walls due to quantum electromagnetic fluctuations. The only modifications are: (i) the speed of light must be substututed by the spin of sound c_s; (ii) the factor $1/2$ must be added, since we have the scalar field of the longitudinal sound wave instead of two polarizations of light. If a is the distance between the plates and A is their area, then the a-dependent contribution to the ground state energy of the quantum liquid at $T = 0$ which follows from the effective theory must be

$$E_C = -\frac{\hbar c_s \pi^2 A}{1440 a^3}. \tag{1}$$

Such microscopic quantities of the quantum liquid as the mass of the atom m and interatomic space a_P do not enter explicitly the Eq.(1): the traditional Casimir force is completely determined by the "fundamental" parameter c_s of the effective scalar field theory.

However, we shall show that the Eq.(1) is not always true. We shall give here an example, where the effective theory is not able to predict the Casimir force, since the microscopic high-energy degrees of freedom become important. In other words the "transPlanckian physics" shows up and the "Planck" energy scale explicitly enters the result. In this situation the Planck scale is physical and cannot be removed by any regularization.

The Eq.(1) gives a finite-size contribution to the energy of quantum liquid. It is inversly proportional to the linear dimension of the sistem, $E_C \propto 1/L$. However, for us it is important that it is not only the finite-size effect, but also the finite-N effect, $E_C \propto N^{-1/3}$, where N is the number of atoms in the liquid in the slab, which is a discrete quantity. Since the main contribution to the vacuum energy is $\propto L^3 \propto N$, the relative correction of order $N^{-4/3}$ means that the Casimir force is the mesoscopic effect. We shall show that in quantum liquids, the essentially larger mesoscopic effects, of the relative order N^{-1}, can be more pronounced. Such a finite-N effect cannot be described by the effective theory dealing with the continuous medium, even if the theory includes the real boundary conditions with the frequency dependence of dielectric permeability.

We shall start with the simplest quantum "liquid" – the one-dimensional Fermi gas – where the mesoscopic Casimir forces can be calculated exactly without invoking any regularization procedure.

Vacuum energy from microscopic theory. We consider the system of N one-dimensional massless fermions, whose continuous energy spectrum is $\omega(p) = cp$, with c playing the role of speed of light. Let us start with the microscopic theory, which is extremely simple: at $T = 0$ the fermions simply occupy all the energy levels below the chemical potential μ. In the continuous limit the total number of particles N and the total energy in the one-dimensional "cavity" of size a are expressed in terms of the Fermi momentum $p_F = \mu/c$ in the following way

$$N = na = a\int_{-p_F}^{p_F} \frac{dp}{2\pi\hbar} = \frac{ap_F}{\pi\hbar}, \tag{2}$$

$$E = \epsilon(n)a = a\int_{-p_F}^{p_F} \frac{dp}{2\pi\hbar} cp = \frac{acp_F^2}{2\pi\hbar} = \frac{\pi}{2}\hbar can^2. \tag{3}$$

Here n is the particle density. The vacuum energy density of this condensed matter as a function of n is $\epsilon(n) = (\pi\hbar c/2)n^2$. The equation of state comes from the thermodynamic identity relating the pressure P and the energy:

$$P = \mu n - \epsilon, \tag{4}$$

where $\mu = d\epsilon/dn = cp_F$ is the chemical potential. In our case $\mu = cp_F$, and one obtains the equation of state for our vacuum

$$P = \epsilon \tag{5}$$

which is conventional for the system of 1+1 relativistic fermions.

Vacuum energy in effective theory. As distinct from the microscopic theory, which deals with bare particles, the effective theory deals with the quasiparticles – fermions living at the level of the chemical potential $\mu = cp_F$. There are 4 different quasiparticles: (i) quasiparticles and quasiholes living in the vicinity of the Fermi point $p_z = +p_F$ have spectrum $\omega_{qp}(p_+) = |\omega(p) - \mu| = c|p_+|$, where $p_+ = p_z - p_F$; (ii) quasiparticles and quasiholes living in the vicinity of the other Fermi point at $p_z = -p_F$ have the spectrum $\omega_{qp}(p_-) = |\omega(p) - \mu| = c|p_-|$, where $p_- = p_z + p_F$. In the effective theory the energy of the system is the energy of the Dirac vacuum $E = -\sum_{p_+} c|p_+| - \sum_{p_-} c|p_-|$. This energy is divergent and requires the cut-off, which is provided by the Fermi-momentum playing the role of the cut-off Planck momentum: $p_F \equiv p_P$. Note that even with this cut-off the energy obtained within the effective theory has a wrong sign, compared with correct microscopic result in Eq.(3).

The difference between the energies obtained in the microscopic and the effective theory approaches becomes important if the gravity is involved, since the energy is the source of the gravitational field. What kind of the vacuum energy is gravitating is the essence of the cosmological constant problem.

Relevant vacuum energy and cosmological constant. Inspection of those condensed matter systems, in which an effective gravity arises as a low energy phenomenon, suggests the possible answer: the vacuum energy density responsible for the cosmological constant is $\tilde{\epsilon} = \epsilon - \mu n$ [9, 10]. This follows from the microscopic physics: the conservation of the particle number N requires that the quantum field theoretical description of the N-body system is given by $\mathcal{H} - \mu\mathcal{N}$, where $\mathcal{H}$ and $\mathcal{N}$ are Hamiltonian and particle number operators in the second quantized form. The energy $\tilde{\epsilon}$ does not depend on the choice of the zero energy level: the shift Δ of the zero energy level for one particle leads to the shift of the chemical potential $\mu \to \mu + \Delta$ and of the total energy $E \to E + N\Delta$, while $\tilde{E} = E - \mu N$ remains invariant. In terms of $\tilde{\epsilon}$ the equation of state of the quantum vacuum is always

$$P = -\tilde{\epsilon}. \tag{6}$$

Though this is obtained using the microscopic theory ($\tilde{\epsilon}$ is not determined within the effective theory), the result does not depend on details of the quantum liquid: it follows from the thermodynamic identity in Eq.(4).

The Eq.(6) is the same as the equation of state of the vacuum in quantum field theory, which follows from the Einstein cosmological term. Thus $\tilde{\epsilon}$ serves as the cosmological constant in the effective gravitational theory. For our vacuum represented by the Fermi gas, this cosmological constant is big, being determined by the "Planck" energy scale, $\tilde{\epsilon} \sim -cp_P^2$. The minus sign is in agreement with the negative energy of the Dirac vacuum in effective theory, and, according to Eq.(6), this corresponds to the positive vacuum pressure: Fermi gas (and the Dirac vacuum too) can be in equilibrium only in the presence of positive external pressure P.

There are, however, quantum liquids which can exist without an external pressure. Liquid ^{3}He and liquid ^{4}He at $T = 0$ are examples. In both of these liquids there is some analog of gravity which arises in the low energy corner. Let us consider the ground state of such quantum liquid, if there is no contact with the environment. In a complete equilibrium the pressure P in the liquid must be zero, since there is no external forces acting on the liquid. Then from the Eq.(6) one automatically obtains that for such equilibrium vacuum at $T = 0$ the cosmological constant in the effective gravity is identically zero, $\tilde{\epsilon} \equiv 0$, without any fine tuning. This means that according to the quantum liquid analogy the stationary equilibrium vacuum is not gravitating (see more details in [10]).

Leakage of vacuum through the wall. Now let us discuss the Casimir effect – the change of the vacuum pressure caused by the finite size effects in the vacuum. We must take into account the discreteness of the spectrum of bare particles or quasiparticles (depending on which theory we use, microscopic or effective) in the slab. Let us start with the microscopic description in terms of bare particles (atoms). We can use two different boundary conditions for particles, which give two kinds of discrete spectrum

$$\omega_k = k\frac{\hbar c\pi}{a}. \tag{7}$$

$$\omega_k = \left(k + \frac{1}{2}\right)\frac{\hbar c\pi}{a}. \tag{8}$$

Eq.(7) corresponds to the "classical spinless" fermions with Dirichlet boundary conditions. The Eq.(8) is for the 1+1 Dirac fermions with no particle current through the wall; this case with the generalization to the $d+1$ fermions has been discussed in [11].

The vacuum is represented by the ground state of the collection of N noninteracting particles in 1D box of size a. The "vacuum" energies for the spinless and the Dirac fermions are correspondingly

$$E(N,a) = \sum_{k=1}^{N} \omega_k = \frac{\hbar c\pi}{2a} N(N+1), \tag{9}$$

$$E(N,a) = \sum_{k=0}^{N-1} \omega_k = \frac{\hbar c\pi}{2a} N^2. \tag{10}$$

To calculate the Casimir force acting on the wall, we must introduce the vacuum on both sides of the wall. Thus let us consider three walls: at $z = 0$, $z = a_1 < a$ and $z = a$. Then we have two slabs with sizes a_1 and $a_2 = a - a_1$, and we can find the force acting on the wall between the two slabs, i.e. at $z = a_1$. We assume the same Neumann boundary conditions at all the walls. But we must allow the particles to transfer between the slabs, otherwise the main force acting on the wall between the slabs will be due to the different bulk pressure in the two slabs. This can be done due to, say, a very small holes (tunnel junctions) in the wall, which do not violate the boundary conditions and do not disturb the particle energy levels, but still allow the particle exchange between the two vacua.

This situation can be compared with the traditional Casimir effect. The force between the conducting plates arises because the electromagnetic fluctuations of the vacuum in the slab are modified due to boundary conditions imposed on electric and magnetic fields. In reality these boundary conditions are applicable only in the low-frequency limit, while the wall is transparent for the high-frequency electromagnetic modes, as well as for the other degrees of freedom of real vacuum (fermionic and bosonic), that can easily penetrate through the conducting wall. In the traditional approach it is assumed that these degrees of freedom, which produce the divergent terms in the vacuum energy, must be cancelled by the proper regularization scheme. That is why, though the dispersion of dielectic permeability does weaken the real Casimir force, nevertheless in the limit of large distances, $a_1 \gg c/\omega_0$, where ω_0 is the characteristic frequency at which the dispersion becomes important, the Casimir force does not depend on how easily the high-energy vacuum leaks through the conducting wall.

We consider here just the opposite limit, when (almost) all the bare particles are totally reflected. This corresponds to the case when the penetration of the high-energy modes of the vacuum through the conducting wall is highly suppressed, and thus one must certainly have the traditional Casimir force. Nevertheless, we shall show that due to the mesoscopic finite-N effects the contribution of the diverging terms to the Casimir effect becomes dominating. They produce highly oscillating vacuum pressure, whose amplitude exceeds by factor $p_P a/\hbar$ the value of the conventional Casimir pressure. For their description the continuous effective low-energy theories are not applicable.

Mesoscopic Casimir force in 1D Fermi gas. The total vacuum energies of the spinless and Dirac fermions in two slabs are

$$E = \frac{\hbar c\pi}{2}\left(\frac{N_1(N_1+1)}{a_1} + \frac{N_2(N_2+1)}{a_2}\right), \qquad (11)$$

$$E = \frac{\hbar c\pi}{2}\left(\frac{N_1^2}{a_1} + \frac{N_2^2}{a_2}\right), \qquad (12)$$

$$N_1 + N_2 = N,\ a_1 + a_2 = a. \qquad (13)$$

Since particles can transfer between the slabs, the global vacuum state in this geometry is obtained by minimization over the discrete particle number N_1 at fixed total number N of particles in the vacuum. If the mesoscopic $1/N$ corrections are ignored, one obtains $N_1 \approx (a_1/a)N$ and $N_2 \approx (a_2/a)N$, and the force acting on the wall between the two vacua is zero.

However, N_1 and N_2 are integer valued, and this leads to mesoscopic fluctuations of the Casimir force. Within a certain range of parameter a_1 there is a global minimum characterized by integers $(N_1,\ N_2)$. In the neighboring intervals of parameters a_1, one has either $(N_1+1,\ N_2-1)$ or $(N_1-1,\ N_2+1)$. The force acting on the wall in the state $(N_1,\ N_2)$ is obtained by variation of $E(N_1, N_2, a_1, a-a_1)$ over a_1 at fixed N_1 and N_2:

$$F_{N_1 N_2 a_1 a_2} = -\frac{dE_{N_1 N_2 a_1 a_2}}{da_1} + \frac{dE_{N_1 N_2 a_1 a_2}}{da_2}\ . \qquad (14)$$

When a_1 increases then at some critical value of a_1, where $E(N_1, N_2, a_1, a_2) = E(N_1+1, N_2-1, a_1, a_2)$, one particle must cross the wall from the right to the left. At this critical value the force acting on the wall changes abruptly (we do not discuss here an interesting physics arising just at the critical values of a_1, where the degeneracy occurs between the states (N_1, N_2) and (N_1+1, N_2-1); at these positions of the wall (or membrane) the particle numbers N_1 and N_2 are undetermined and are actually fractional due to the quantum tunneling between the slabs [12]). Using for example the spectrum in Eq.(12) one obtains for the jump of the Casimir force:

$$F_{N_1\pm 1, N_2\mp 1} - F_{N_1,N_2} \approx \pm\frac{\hbar c\pi N}{a_1 a_2}. \qquad (15)$$

If $a_1 \ll a$ the amplitude of the mesoscopic force

$$|\Delta F_{\rm meso}| = \frac{\hbar c\pi n}{a_1} = \frac{\hbar c\pi n^2}{N_1} \equiv \frac{c p_P}{a_1}\ . \qquad (16)$$

It is by factor $1/N_1 = (\pi\hbar/a_1 p_F)^3 \equiv (\pi\hbar/a_1 p_P)^3$ smaller than the bulk vacuum energy density in Eq.(3). On the other hand it is by the factor $p_F a_1 \equiv p_P a_1$ larger than the traditional Casimir pressure, which in 1D case is $P_C \sim \hbar c/a_1^2$. The divergent term which linearly depends on the Planck momentum cutoff p_P as in Eq.(16) has been revealed in many different calculations (see e.g. [6]), and attempts have made to invent the regularization scheme which would cancel the divergent contribution.

Mesoscopic Casimir forces in a general condensed matter system. Eq.(16) for the amplitude of mesoscopic fluctuations of the vacuum pressure can be generalized to any dimension. The mesoscopic random pressure comes from the discrete nature of the quantum vacuum in quantum liquids. When the volume V_1 of the vessel changes continuously, the equilibium number N_1 of particles changes in step-wise manner. This results in abrupt changes of pressure at some critical values of V_1:

$$P_{\rm meso} \sim P_{N_1\pm 1} - P_{N_1} = \pm\frac{dP}{dN_1} = \pm\frac{mc_s^2}{V_1} \equiv \pm\frac{cp_P}{V_1}, \qquad (17)$$

The mesoscopic pressure is determined by microscopic physics, and thus such microscopic quantity as the mass m of the atom, the "Planck mass", enters this force.

For the pair correlated systems, such as Fermi superfluids with finite gap in the energy spectrum, the amplitude must be twice larger. This is because the jumps in pressure occurs when two particles (the Cooper pair) tunnel through the junction, $\Delta N = \pm 2$.

For the spherical shell of radius a immersed in the quantum liquid the mesoscopic pressure is

$$P_{\rm meso} \sim \pm \frac{3mc_s^2}{4\pi a^3} \equiv \pm \frac{3p_P c}{4\pi a^3}. \qquad (18)$$

Discussion. Let us compare the mesoscopic vacuum pressure in Eq.(18) with the traditional Casimir pressure obtained within the effective theories for the same spherical shell geometry. In case of the original Casimir effect the effective theory is quantum electrodynamics. In superfluid ^{4}He this is the low-frequency quantum hydrodynamics, which is equivalent to the relativistic scalar field theory. The sound wave modes with linear ("relativistic") spectrum play the role of the relativistic massless scalar field with Neumann boundary conditions, corresponding to the (almost) vanishing current through the wall (let us recall that there must be some leakage through the shell to provide the equal bulk pressure on both sides of the shell).

If we believe in the traditional regularization schemes which cancel out the ultraviolet divergence, then from the effective scalar field theory one must obtain the Casimir pressure $P_C = -dE_C/dV = K\hbar c_s/8\pi a^4$, where $K = -0.4439$ for the Neumann boundary conditions; $K = 0.005639$ for the Dirichlet boundary conditions [6]. However, at least in our case, the result obtained within the effective theory is not correct: the real Casimir pressure in Eq.(18) is produced by the finite-N effect. It essentially depends on the Planck cut-off parameter, i.e. it cannot be determined by the effective theory; it is much bigger, by the factor $p_P a/\hbar$, than the traditional Casimir pressure; and it is highly oscillating. The regularization of these oscillations by, say, averaging over many measurements; by noise; or due to quantum or thermal fluctuations of the shell; etc., depend on the concrete physical conditions of the experiment.

This shows that in some cases the Casimir vacuum pressure is not within the responsibility of the effective theory, and the microscopic (transPlanckian) physics must be evoked. If two systems have the same low-energy behavior and are described by the same effective theory, they do not necessarily experience the same Casimir effect. The result depends on many factors: discrete nature of the quantum vacuum; ability of the vacuum to penetrate through the boundaries; dispersion relation at high frequency, etc. It is not excluded that even the original electromagnetic Casimir effect is renormalized by high-energy modes.

Of course, the extreme limit of almost impenetrable wall, which we considered, is not applicable to the original (electromagnetic) Casimir effect, where the overwhelming part of the fermionic and bosonic vacua easily penetrates the conducting walls, and where the mesoscopic fluctuations must be small. But are they negligibly small? In any case our example shows that the cut-off problem is not the mathematical, but the physical one, and the physics dictates the proper regularization scheme or the proper choice of the cut-off parameters.

The dependence of low-energy effects on physics beyond the effective theory was discussed also in connection with the Chern-Simons terms violating Lorentz and CPT symmetries [13, 14]. Quantum liquids provide an example of finite system where the "transPlanckian" microscopic physics determines the coefficient in front of the Chern-Simons term [9, 15], which remains ambiguous within the effective theory.

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