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Fermion zero modes in Painlevé-Gullstrand black hole

G. E. Volovik

Low Temperature Laboratory, Helsinki University of Technology, FIN-02015 HUT, Finland
and
L. D. Landau Institute for Theoretical Physics RAS, 117940 Moscow, Russia

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Painlevé-Gullstrand metric of the black hole allows to discuss the fermion zero modes inside the hole. The statistical mechanics of the fermionic microstates can be responsible for the black hole thermodynamics. These fermion zero modes also lead to quantization of the horizon area.

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1. Introduction. In general relativity there are different nonequivalent metrics $g_{\mu\nu}$, which describe the same gravitational object. Though they can be obtained from each other by coordinate transformations, in the presence of the event horizon they produce the nonequivalent quantum vacua. Among the other metrics used for description of the black hole, the Painlevé-Gullstrand metric [1] has many advantages [2, 3] and now it acquires more popularity (see e.g. [4]; on extension of Painlevé-Gullstrand metric to the rotating black hole see [5]). Also such a stationary but not static metric naturally arises in the condensed matter analogs of gravity [6 – 10]. Here using the Painlevé-Gullstrand metric we consider the structure of the low energy fermionic microstates in the interior of the black hole and their contribution to the black hole thermodynamics.

The interval in the Painlevé-Gullstrand space-time is

$$ds^2 = -\left(c^2 - v_{\rm s}^2(r)\right) dt^2 - 2v_{\rm s}(r) dr dt + dr^2 + r^2 d\Omega^2. \quad (1)$$

In the acoustic black and white holes, $v_{\rm s}(r)$ is the radial velocity of the fluid, which produces the effective metric for acoustic waves– phonons – propagating in the liquid [6 – 8]. For the gravitational field produced by the point source of mass M, the function $v_{\rm s}(r)$ has the form

$$v_{\rm s}(r) = \pm c\sqrt{r_h/r}, \quad r_h = 2MG, \quad (2)$$

where r_h is the radius of the horizon; G is the Newton gravitational constant; $c = \hbar = 1$. The Painlevé-Gullstrand metric breaks the time reversal symmetry: the time reversal operation transforms black hole to the white hole (see also [3]). The minus sign in Eq.(2) gives the metric for the black hole. In the fluid analog of gravity this corresponds to the liquid flowing inward. The plus sign characterizes the white hole and correspondingly the flow outward in the fluid analogy. The time reversal operation reverses the direction of flow.

The Painlevé-Gullstrand metric describes the space-time both in exterior and interior regions. This space-time, though not static, is stationary. That is why the energy in the interior region is well determined. Moreover, as distinct from the Schwarzschild metric, the particle energy spectrum $E(\mathbf{p})$ (the solution of equation $g^{\mu\nu}p_\mu p_\nu + m^2 = 0$ where $p_0 = -E$) is well defined for any value of the momentum $\mathbf{p}$. This allows us to determine the ground state (vacuum) of the Standard Model in the interior region and the thermal states – the black hole matter. We consider here only the fermionic vacuum of the Standard Model, assuming that the Standard Model is an effective theory and thus the bosonic fields are the collective modes of the fermionic vacuum.

2. Fermi surface for Standard Model fermions inside horizon. We shall see that the main contribution to the thermodynamics of the black hole comes from the very short wave lengths of order of the Planck length. That is why the ultraviolet cut-off must be introduced. We introduce the cut-off using the nonlinear dispersion of the particle spectrum in the ultrarelativistic region, which violates the Lorentz symmetry at short distances. Such nonlinear dispersion of the particle spectrum is now frequently used both in the black hole physics and cosmology [11 – 12]. We shall use the superluminal dispersion, for which the velocity of the particle becomes superluminal at very high momentum. For the simplest superluminal dispersion the energy spectrum of the fermionic particle in the Painlevé-Gullstrand metric becomes

$$E = p_r v_{\rm s} \pm c\sqrt{p^2 + p^4/p_0^2} \quad (3)$$

where p_r is the radial momentum of the particle; p_0 is plays the role of the cut-off momentum which is somewhat less than the Planck momentum $p_{\rm Planck}$; and we neglected all the masses of the Standard Model fermions, since they are much less than the characteristic energy scales.

Because of the possibility of the superluminal propagation, the surface $r = r_h$ is not the true horizon. This surface marks the boundary of the ergoregion: at $r < r_h$ particles with positive square root in Eq.(3) can acquire negative energy. As a result, at $r < r_h$ the Fermi surface appears – the surface in the 3D momentum space, where the energy of particles is zero, $E(\mathbf{p}) = 0$. For the spectrum in Eq.(3) the surface is given by equation, which expresses the radial momentum in terms of the transverse momentum $p_\perp$:

$$p_r^2(p_\perp) = \frac{1}{2}p_0^2(v_{\rm s}^2 - 1) - p_\perp^2 \pm \sqrt{\frac{1}{4}p_0^4(v_{\rm s}^2 - 1)^2 - p_0^2 v_{\rm s}^2 p_\perp^2} \tag{4}$$

This surface exists at each point $\mathbf{r}$ within the horizon (ergosurface), where $v_{\rm s}^2 > 1$. It exists only in the restricted range of the transverse momenta, with the restriction provided by the cut-off parameter p_0:

$$p_\perp < \frac{1}{2}p_0\left|v_{\rm s} - \frac{1}{v_{\rm s}}\right|. \tag{5}$$

This means that the Fermi surface is a closed surface in the 3D momentum space $\mathbf{p}$.

The Fermi surface provides the finite density of fermionic states (DOS) at $E = 0$

$$N(E=0) = N_F \sum\nolimits_{\mathbf{p},\mathbf{r}} \delta(E(\mathbf{p})) = \tag{6}$$

$$= \frac{4\pi N_F}{(2\pi)^3}\int_0^{r_h} r^2 dr \int d^3p\, \delta\left(p_r v_{\rm s} + c\sqrt{p^2 + \frac{p^4}{p_0^2}}\right) = \tag{7}$$

$$= \frac{N_F}{\pi}\int_0^{r_h} r^2 dr \int_0^{p_\perp^2(r)} \frac{d(p_\perp^2)}{|v_G|}. \tag{8}$$

Here $N_F = 16 N_g$ is the number of the massless chiral fermionic species in the Standard Model with N_g generations; v_G is the radial component of the group velocity of particles at the Fermi surface:

$$v_G(E=0) = \frac{dE}{dp_r} = \mp\sqrt{\left(v_{\rm s} - \frac{1}{v_{\rm s}}\right)^2 - 4\frac{p_\perp^2}{p_0^2}}. \tag{9}$$

Integration over $p_\perp^2$ in Eq.(8) gives for the density of states

$$N(E=0) = \frac{N_F p_0^2}{\pi}\int_0^{r_h} r^2 dr \left|v_{\rm s} - \frac{1}{v_{\rm s}}\right| = \frac{4N_F}{35\pi}p_0^2 r_h^3. \tag{10}$$

The main contribution to DOS and thus to the thermodynamics comes from the momenta p comparable with the cut-off momentum p_0. That is why all the masses of fermions were neglected.

The DOS $N(E=0)$ determines the thermodynamics of the black hole matter at $T \neq 0$. The thermal energy $\mathcal{E}(T)$ and entropy $\mathcal{S}(T)$ carried by the Standard Model fermions in the interior of the black hole with nonzero temperature is

$$\mathcal{E}(T) = N(0)\int dE E f(E/T) = \frac{\pi^2}{6}N(0)T^2, \qquad \mathcal{S}(T) = \frac{\pi^2}{3}N(0)T, \tag{11}$$

where $f(x) = 1/(e^x + 1)$ is the Fermi distribution function.

We considered the large temperature as compared to the Hawking temperature $T_{\rm H} = \hbar c/4\pi r_h$ [13]. At lower energies the discrete nature of the spectrum of the Standard Model fermions bound to the black hole becomes important.

At $T \sim T_{\rm H}$ the entropy becomes of order

$$\mathcal{S}(T \sim T_{\rm H}) \sim N_F p_0^2 r_h^2 \tag{12}$$

The cutoff momentum p_0 can be expressed in terms of the effective gravitational constant G, which is determined by the same cutoff according to the Sakharov induced gravity [14]. The effective action for gravity is obtained by integration over vacuum fermions, and thus all fermionic species must add to produce the effective Newton constant: $G^{-1} \sim p_0^2 N_F$. Thus the thermal entropy in Eq.(12) is scaled as G^{-1}, i.e. $\mathcal{S}(T \sim T_{\rm H}) \sim r_h^2/G$. The same occurs with the Bekenstein-Hawking entropy of the black hole, $S_{\rm BH} = \pi r_h^2/G$. As it was first shown by Jacobson, $S_{\rm BH}$ is renormalized by the same quantum fluctuations as G^{-1}, and thus is proportional to G^{-1} [15]. Thus the thermal entropy of the Standard Model fermionic microstates within the black hole at $T \sim T_{\rm H}$ has the same behavior and the same order of magnitude as the Bekenstein-Hawking entropy of the black hole.

3. Discrete energy levels inside horizon. Now we proceed to the low energy, where the quantization is important and gives discrete energy levels for the Standard Model fermions within the horizon. Since the momenta of particles are large compared to the size of the horizon, one can use the quasiclassical approximation for the radial motion and the Bohr-Sommerfeld quantization rule. We consider here the low energy states whose energy E is much less than the characteristic energy scale of the Fermi liquid: $E \ll p_0 c$. In this limit the classical trajectories, which determine the Bohr-Sommerfeld quantization, can be obtained by perturbation theory. Let us start with the zero order trajectories, i.e. trajectories with $E = 0$. After quantization of the azimuthal motion, one obtains the following dependence of the radial momentum p_r on r, which determines the classical trajectories along the radius at a given value of the angular momentum L (compare this with Eq.(4)):

$$p_r^2(r, E=0, L) = \frac{1}{2}p_0^2(v_s^2 - 1) - \\ - \frac{L^2}{r^2} \pm \sqrt{\frac{1}{4}p_0^4(v_s^2-1)^2 - p_0^2 v_s^2 \frac{L^2}{r^2}}. \quad (13)$$

Since for typical bound states one has $L \gg 1$, the difference between expressions $L(L+1)$, $(L+\frac{1}{2})^2$ and L^2 is not important.

The trajectories in Eq.(13) are closed: there are two turning points on each trajectory. Particle moves back and forth between the zeroes r_1 and r_2 of the square root in the right hand side of Eq.(13):

$$\left| v_s(r_{1,2}) - \frac{1}{v_s(r_{1,2})} \right| = \frac{2L}{p_0 r_{1,2}} . \quad (14)$$

At the turning points the group velocity of the particle v_G in Eq.(9) becomes zero and changes sign to the opposite. This is very similar to Andreev reflection [16]: the velocity changes sign after reflection while the momentum p_r does not.

For nonzero but small energy, $E \ll p_0 c$, the trajectories are obtained by perturbation theory. The first order correction gives

$$p_r(r,E,L) = p_r(r, E=0, L) + \frac{dp_r}{dE}E = \\ = p_r(r,E=0,L) + \frac{E}{v_G(E=0)}. \quad (15)$$

The Bohr-Sommerfeld quantization gives

$$2\pi(n_r + \gamma(L)) = \oint dr p_r(r, E=0, L) + E \oint \frac{dr}{v_G}, \quad (16)$$

where n_r is the radial quantum number; and $\gamma(L)$ is the parameter of order unity, which is not determined within this quantization scheme. Numerical integration of $\oint dr p_r(r, E=0, L)$ shows that it is very close to the following equation[1)]

$$\oint dr p_r(r, E=0, L) = \pm \frac{3\sqrt{3}\pi}{2}\left(L_{max} - L\right), \\ L_{max} = \frac{p_0 r_h}{3\sqrt{3}}. \quad (17)$$

As a result one obtains the following equidistant energy levels for each L:

$$E = \omega_0(L)\left(n_r + \gamma(L) \mp \frac{3\sqrt{3}}{4}\left(L_{max} - L\right)\right), \quad (18)$$

$$\frac{\pi}{\omega_0(L)} = \int_{r_1}^{r_2} dr \frac{1}{\sqrt{\left(v_s - 1/v_s\right)^2 - 4L^2/r^2 p_0^2}}. \quad (19)$$

In two limiting cases the interlevel distance:

$$\omega_0(L \ll p_0 r_h) \approx \frac{\pi \hbar c}{r_h \ln(p_0 r_h/L)} = T_{\rm H}\frac{4\pi^2}{\ln\left(p_0 r_h/L\right)}, \\ \omega_0(L \to L_{max}) = \frac{3\hbar c}{r_h} = 12\pi T_{\rm H}. \quad (20)$$

Thus for each spherical harmonics L, L_z there are bound states in the black hole interior whose energy as a function of the radial quantum number n_r crosses zero energy level at $n_r \approx \pm(3\sqrt{3}/4)\left(L_{max} - L\right)$. These are the branches of the fermion zero modes. The total number of such branches is

$$N_{\rm zm} = 2N_F \sum_{L<L_{\rm max}}(2L+1) \approx \\ \approx 2N_F L_{\rm max}^2 = \frac{2}{27}N_F p_0^2 r_h^2 \sim \frac{A}{G}, \quad (21)$$

where $A = 4\pi r_h^2$ is the area of the black hole horizon. The estimation of the density of states remains the same as in Eq.(10) which was obtained within the the Fermi-surface approach:

$$N(0) = 2N_F \sum_L \frac{2L+1}{\omega_0(L)} = 2N_F \int_0^\infty d(L^2)\frac{1}{\omega_0(L)} = \quad (22)$$

$$= \frac{N_F p_0^2}{\pi}\int_0^{r_h} r^2 dr \left| v_s - \frac{1}{v_s}\right| = \frac{4N_F}{35\pi}p_0^2 r_h^3. \quad (23)$$

4. Fermion zero modes: vortex vs black hole. The energy spectrum of the ultrarelativistic fermions within the black hole in Eq.(18) resembles the spectrum of fermionic bound state within the core of vortices in Fermi superfluids and superconductors: The energy levels are also equidistant there [17]. The energy spectrum of fermion zero modes in the vortex core depends on two quantum number appropriate for the states within the linear oblect, linear and angular momenta along the vortex axis:

$$E(L_z, p_z) = \omega_0(p_z)(L_z + \gamma), \quad (24)$$

where parameter γ is either 0 or 1/2 depending on the type of the vortex (see review paper [8]). For vortices with $\gamma = 0$ the energy levels with $L_z = 0$ have exactly zero energy. For such a vortex the entropy is nonzero

1)I am indebted to V.B. Eltsov for these calculations.

even at $T = 0$. Each of the states with $E = 0$ can be either free or occupied by fermion. This gives the zero temperature entropy $\ln 2$ per each $E = 0$ state with given p_z. Thus the total entropy of the vortex at $T = 0$ is proportional to the length l of the vortex line:

$$\mathcal{S}(T=0) = N_{\rm zm} \ln 2, \ N_{\rm zm} \propto p_0 l. \tag{25}$$

Here $p_0 = p_F$ is the Fermi momentum of the Fermi superfluid or superconductor; it plays the role of the cut-off parameter. It is interesting that, as in the case of the black hole, the wavelengths of fermions comprising the fermion zero modes in the vortex core are much less than the core size. This allows us to use the quasiclassical theory. However, even within the quasiclassical theory one can, using the symmetry or other arguments, find the value of the phase shift γ in the Bohr-Sommerfeld quantization scheme [8] and predict for which vortex the system of the equidistant levels of fermions contains the states with exactly zero energy.

For the fermionic states bound to the black hole the parameter $\gamma(L)$ in Eq.(18) is still unknown. That is why one cannot say whether the system of the equidistant levels contains the level with zero energy or not. If yes, then each state with zero energy contributes the entropy $\ln 2$; the total entropy provided by the fermion zero mode at $T = 0$ is

$$\mathcal{S}(T=0) = N_{\rm zm} \ln 2. \tag{26}$$

5. Discussion. From Eq.(21) it follows that for the Painlevé-Gullstrand black hole the area of the black hole horizon is expressed in terms of the integer valued quantity:

$$A p^2_{\rm Planck} = \sigma \mathcal{N}, \tag{27}$$

where $\mathcal{N}$ is the number of fermion zero modes within the black hole: $\mathcal{N} = N_{\rm zm}$; and σ is of order unity. This formula with different values of the parameter σ was discussed in many modern theories of black holes (see [18, 19] and references therein). It was interpreted as quantization of the horizon area, with $\mathcal{N}$ being the quantum number which characterizes the black hole as an 'atom' [18]. If one uses $\sigma = 4\ln 2$ as in Ref. [20], one obtains the Bekenstein-Hawking entropy in Eq.(26). $\mathcal{N}$ was also interpreted as the number of 'constituents' of the black hole interior – the 'gravitational atoms' [21]. In our case both interpretations are applicable, though with some reservation.

The quantization of area in Eq.(27) usually suggests that the spacing between the levels is uniform and is on the order of $dM/d\mathcal{N} \propto E^2_{\rm Planck}/M$ [18]. This is in agreement with the Eq.(20) for the interlevel distance ω_0 in the fermionic spectrum. However, the quantization in terms of the number of fermion zero modes suggests another possible interpretation:

The area A of the black hole is a continuous parameter. When it changes, the number of fermion zero modes $N_{\rm zm}(A)$ as a function of A changes in step-wise manner at some critical values of A. This is what happens, say, in the integer quantum Hall effect, where the integer topological charge $\mathcal{N}$ of the qauntum vacuum as a function of external parameters has plateaus. If the external parameter is the magnetic field B, then $\mathcal{N}(B)$ and the Hall conductivity $\sigma_{xy}(B) = (e^2/h)\mathcal{N}(B)$ change abruptly when the critical values of the magnetic field B are crossed. Similar behavior of the topological charge $\mathcal{N}$ of the quantum vacuum occurs in other quasi-2D fermionic systems too, e.g. the momentum-space topological charge $\mathcal{N}$ of the film of a quantum liquid is a step-wise function of the continuous parameter – the thickness of the film (see Chapter 9 in Ref. [22]).

$\mathcal{N}$ in Eq.(27) can be also related to the number of 'constituents' as suggested in Ref. [21]. According to the Fermi liquid description, the number of thermal fermions in the Fermi liquid at temperature T is $N_{thermal} \sim N(0)T \sim A/G$. According to Eq.(11) each fermion carries energy of order T. At $T \sim T_{\rm H}$ their total thermal energy is on the order of the mass M of the black hole, and they carry the thermal entropy of the order of the Hawking-Bekenstein entropy $S_{\rm BH}$. Assuming that the whole mass M of the black hole comes from thermal fermions within the horizon, one has $M = E + 3pV$, where $V = (4\pi/3)r_h^3$ is the volume within the horizon and p the pressure of the fermionic system. Using the equation of state of thermal fermions forming the Fermi surface, $E = ST/2 = pV$, which follows from Eq.(11), one obtains the correct relation between the mass, Hawking temperature and Bekenstein entropy of the black hole:

$$M = E + 3pV = 4E = 2TS. \tag{28}$$

The Bekenstein-Hawking entropy, $S = \pi r_h^2$, and the Hawking temperature, $T_{\rm H} = 1/4\pi r_h$, are reproduced if the relation between the cut-off p_0 and the Newton constant is $G^{-1} = N_F p_0^2/105\pi$. Thus the black hole fermionic matter at the Hawking temperature can provide the mass and the entropy of the black hole: fermions thermally excited within the horizon can serve as 'constituents'.

These constituents do not actually represent the 'gravitational atoms' which form the quantum vacuum and give rise to the phenomenon of gravitation according to Ref. [23]. These are conventional elementary particles of the Standard Model (quarks and leptons) who are excited within the black hole. Their contribution is essential even at the temperature as low as $T_{\rm H}$, because of the huge DOS within the black hole.

On the other hand, these constituents have little to do with the matter absorbed by the black hole during its formation. The fermions, which form the matter within the black holes, are all the fermions of the Standard Model, quarks and leptons, which are highly ultrarelativistic. The black hole metric emerging after collapse perturbs significantly the spectrum of Standard Model fermions, so that the Fermi surface appears which provides a huge DOS at zero energy. Essential part of these fermions have momenta of order of Planck scale; for them the effective gravitational theory probably is not applicable. In this sense these fermions are close to 'gravitational atoms' of trans-Planckian physics.

We considered the vacuum of the Standard Model fermions and their thermal states as viewed in the Painlevé-Gullstrand metric. This vacuum is substantially different from the vacuum state as viewed by comoving observer. The reconstruction of the vacuum within the black hole involves the Planck energy scale, and results depend on the cut-off procedure. The cut-off procedure, on the other hand, depends on the coordinate system used and it assumes the existence of the preferred coordinate frame at high energy. That is why the vacuum structure depends on the coordinate system.

Since the Planck energy scale is involved, it is not clear whether the traditional description of the black hole is applicable. Moreover, the stability of this new vacuum is not guaranteed. In most of those condensed matter systems, where the analog of the event horizon is possible, the vacuum becomes unstable in the presence of horizon, i.e. the quantum vacuum of the condensed matter resists to the formation of a horizon [8, 24]. Also the huge density of states may generate the symmetry breaking in the black hole interior, as it happens in the core of vortices [25] and cosmic strings [26].

Even if the black hole survives under such reconstruction of the Standard Model vacuum, there is another problem to be solved. When thermal states of the fermionic black-hole matter are considered, their energy and pressure must serve as a source of gravitational field according to (maybe somewhat modified) Einstein equations. This will change the field $v_{\rm s}$ which enters the black hole metric and thus the energy spectrum will be modified, but will remain equidistant for each L.

In conclusion, we considered the statistical mechanics of fermionic microstates – the Standard Model fermions – in the interior of a black hole. Fermion zero modes give the correct dependence of the entropy of the Painlevé-Gullstrand black hole on the area of the hole, on the number of fermionic species, and on the Planck cut-off parameter. They also lead to quantization of the horizon area. That is why fermion zero modes can be the true microstates, that are responsible for the black hole thermodynamics.

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1. P. Painlevé, C. R. Acad. Sci. (Paris) **173**, 677 (1921); A. Gullstrand, Arkiv. Mat. Astron. Fys. **16**(8), 1 (1922).
2. K. Martel and E. Poisson, Am. J. Phys. **69**, 476 (2001).
3. R. Schützhold, gr-qc/0011047.
4. M. K. Parikh and F. Wilczek, Phys. Rev. Lett. **85**, 5042 (2000).
5. C. Doran, Phys. Rev. **D61**, 067503 (2000).
6. W. G. Unruh, Phys. Rev. Lett. **46**, 1351 (1981); Phys. Rev. **D51**, 2827 (1995).
7. M. Visser, Class. Quantum Grav. **15**, 1767 (1998).
8. G. E. Volovik, gr-qc/0005091.
9. T. Jacobson and G. E. Volovik, Phys. Rev. **D58**, 064021 (1998).
10. M. Stone, cond-mat/0012316.
11. S. Corley and T. Jacobson, Phys. Rev. **D54**, 1568-1586 (1996); S. Corley, Phys. Rev. **D57**, 6280 (1998); S. Corley, and T. Jacobson, Phys. Rev. **D59**, 124011 (1999).
12. A. Starobinsky, Pis'ma ZhETF, **73**, 415 (2001).
13. S. W. Hawking, Nature **248**, 30 (1974).
14. A. D. Sakharov, Sov. Phys. Dokl. **12**, 1040 (1968).
15. T. Jacobson, gr-qc/9404039.
16. A. F. Andreev, Sov. Phys. JETP **19**, 1228 (1964).
17. C. Caroli, P.G. de Gennes, and J. Matricon, Phys. Lett. **9**, 307 (1964).
18. J. D. Bekenstein, in: Proc. of The Eighth Marcel Grossman Meeting, Ed. Tsvi Piran, Singapore, World Scientific, 1999; gr-qc/9710076.
19. H. A. Kastrup, Phys. Lett. **B413**, 267(1997).
20. V. F. Mukhanov, JETP Lett. **44**, 63 (1986).
21. P. O. Mazur, in: Proc. of The Eighth Marcel Grossman Meeting, Ed. Tsvi Piran, Singapore, World Scientific, 1999; hep-th/9712208.
22. G. E. Volovik, *Exotic properties of superfluid* ^{3}*He*, World Scientific, Singapore, 1992.
23. P. O. Mazur, AIP conf. proceedings **415**, Eds. G. Eigen, P. Osland, and B. Stugu, Woodbury, AIP, 1997; hep-th/9708133.
24. S. Liberati, S. Sonego and M. Visser, Class. Quant. Grav. **17**, 2903 (2000).
25. Yu. G. Makhlin and G. E. Volovik, JETP Lett. **62**, 737 (1995).
26. S. G. Naculich, Phys. Rev. Lett. **75**, 998-1001, (1995).