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Classical and quantum regimes of the superfluid turbulence

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We argue that turbulence in superfluids is governed by two dimensionless parameters. One of them is the intrinsic parameter q which characterizes the friction forces acting on a vortex moving with respect to the heat bath, with q^{-1} playing the same role as the Reynolds number $\mathrm{Re} = UR/\nu$ in classical hydrodynamics. It marks the transition between the "laminar" and turbulent regimes of vortex dynamics. The developed turbulence described by Kolmogorov cascade occurs when $\mathrm{Re} \gg 1$ in classical hydrodynamics, and $q \ll 1$ in the superfluid hydrodynamics. Another parameter of the superfluid turbulence is the superfluid Reynolds number $\mathrm{Re}_s = UR/\kappa$, which contains the circulation quantum κ characterizing quantized vorticity in superfluids. This parameter may regulate the crossover or transition between two classes of superfluid turbulence: (i) the classical regime of Kolmogorov cascade where vortices are locally polarized and the quantization of vorticity is not important; (ii) the quantum Vinen turbulence whose properties are determined by the quantization of vorticity. The phase diagram of the dynamical vortex states is suggested.

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1. Introduction. The hydrodynamics of superfluid liquid exhibits new features with respect to conventional classical hydrodynamics, which are important when the turbulence in superfluids is considered [1].

(i) The superfluid liquid consists of two mutually penetrating components – the frictionless superfluid component and the viscous normal component. That is why different types of turbulent motion are possible depending on whether the normal and superfluid components move together or separately. Here we are interested in the most simple case when the dynamics of the normal component can be neglected. This occurs, for example, in superfluid phases of ^{3}He where the normal component is so viscous that it is practically clamped by the container walls. Its role is to provide the preferred reference frame, where the normal component and thus the heat bath are at rest. The turbulence in the superfluid component with the normal component at rest is referred to as the superfluid turbulence.

(ii) The important feature of the superfluid turbulence is that the vorticity of the superfluid component is quantized in terms of the elementary circulation quantum κ. So the superfluid turbulence is the chaotic motion of well determined and well separated vortex filaments [1]. Using this we can simulate the main ingredient of the classical turbulence – the chaotic dynamics of the vortex degrees of freedom of the liquid.

(iii) The further simplification comes from the fact that the dissipation of the vortex motion is not due to the viscosity term in the Navier-Stokes equation which is proportional to the velocity gradients $\nabla^2\mathbf{v}$ in classical liquid, but due to the friction force acting on the vortex when it moves with respect to the heat bath (the normal component). This force is proportional to velocity of the vortex, and thus the complications resulting from the $\nabla^2\mathbf{v}$ term are avoided.

Here we discuss how these new features could influence the superfluid turbulence.

2. Coarse-grained hydrodynamic equation. The coarse-grained hydrodynamic equation for the superfluid vorticity is obtained from the Euler equation for the superfluid velocity $\mathbf{v} \equiv \mathbf{v}_s$ after averaging over the vortex lines (see the review paper [2]). Instead of the Navier-Stokes equation with $\nabla^2\mathbf{v}$ term one has

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\mu = \mathbf{v}\times\boldsymbol{\omega} - \quad (1)$$

$$-\alpha'(\mathbf{v}-\mathbf{v}_n)\times\boldsymbol{\omega} + \alpha\,\hat{\boldsymbol{\omega}}\times(\boldsymbol{\omega}\times(\mathbf{v}-\mathbf{v}_n))\ . \quad (2)$$

Here $\mathbf{v}_n$ is the velocity of the normal component; $\boldsymbol{\omega} = \nabla\times\mathbf{v}$ is the superfluid vorticity; $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}/\omega$; and dimensionless parameters α' and α come from the reactive and dissipative forces acting on a vortex when it moves with respect to the normal component. These parameters are very similar to the Hall resistivity ρ_{xy} and ρ_{xx} in the Hall effect. For vortices in fermionic systems (superfluid ^{3}He and superconductors) they were calculated by Kopnin [3], and measured in ^{3}He-B in the broad temperature

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range by Bevan et al. [4] (see also [5], where these parameters are discussed in terms of the chiral anomaly).

The terms in Eq.(1) are invariant with respect to the transformation $\mathbf{v} \to \mathbf{v}(\mathbf{r}-\mathbf{u}t)+\mathbf{u}$ as in classical hydrodynamics. However, the terms in Eq.(2) are not invariant under this transformation, since there is the preferred reference frame in which the normal component is at rest. They are invariant under the full Galilean transformation when the normal component is also involved: $\mathbf{v} \to \mathbf{v}(\mathbf{r}-\mathbf{u}t)+\mathbf{u}$ and $\mathbf{v}_\mathrm{n} \to \mathbf{v}_\mathrm{n}+\mathbf{u}$.

Further we shall work in the frame where $\mathbf{v}_\mathrm{n} = 0$, but we must remember that this frame is unique. In this frame the equation for superfluid hydrodynamics is simplified:

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\mu = (1-\alpha')\mathbf{v}\times\boldsymbol{\omega} + \alpha\,\hat{\boldsymbol{\omega}}\times(\boldsymbol{\omega}\times\mathbf{v}). \qquad (3)$$

After rescaling the time, $\tilde{t} = (1-\alpha')t$, one obtains equation which depends on a single parameter $q = \alpha/(1-\alpha')$:

$$\frac{\partial \mathbf{v}}{\partial \tilde{t}} + \nabla\tilde{\mu} = \mathbf{v}\times\boldsymbol{\omega} + q\,\hat{\boldsymbol{\omega}}\times(\boldsymbol{\omega}\times\mathbf{v}). \qquad (4)$$

Now the first three terms together are the same as inertial terms in classical hydrodynamics. They satisfy the modified Galilean invariance (in fact the transformation below changes the chemical potential, but this does not influence the vortex degrees of freedom which are important for the phenomenon of turbulence):

$$\mathbf{v}(\tilde{t},\mathbf{r}) \to \mathbf{v}(\tilde{t},\mathbf{r}-\mathbf{u}\tilde{t}) + \mathbf{u}. \qquad (5)$$

On the contrary the dissipative last term with the factor q in Eq.(4) is not invariant under this transformation. This is in contrast to the conventional liquid where the whole Navier-Stokes equation which contains viscosity

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\mu = \mathbf{v}\times\boldsymbol{\omega} + \nu\nabla^2\mathbf{v}, \qquad (6)$$

is Galilean invariant, and where there is no preferred reference frame.

Such a difference between the dissipative last terms in Eqs. (6) and (4) is very important:

(1) The role of the Reynolds number, which characterizes the ratio of inertial and dissipative terms in hydrodynamic equations, in the superfluid turbulence is played by the intrinsic parameter $1/q$. This parameter does not depend on the characteristic velocity U and size R of the large-scale flow as distinct from the conventional Reynolds number $\mathrm{Re} = RU/\nu$ in classical viscous hydrodynamics. That is why the turbulent regime occurs only at $1/q > 1$ even if vortices are injected to the superfluid which moves with large velocity U. This rather unexpected result was obtained in recent experiments with superfluid ^{3}He-B [6].

(2) In the conventional turbulence the large-scale velocity U is always understood as the largest characteristic velocity difference in the inhomogeneous flow of classical liquid [7]. In the two-fluid system the velocity U is the large-scale velocity of superfluid component with respect to the normal component, and this velocity (the so-called counterflow velocity) can be completely homogeneous.

(3) As a result, as distinct from the classical hydrodynamics, the energy dissipation which is produced by the last term in Eq.(4) depends explicitly on U:

$$\epsilon = -\dot{E} = -\langle \mathbf{v}\cdot\frac{\partial \mathbf{v}}{\partial \tilde{t}}\rangle = -q\langle \mathbf{v}\cdot(\hat{\boldsymbol{\omega}}\times(\boldsymbol{\omega}\times\mathbf{v}))\rangle \sim q\omega U^2. \qquad (7)$$

(4) The onset of the superfluid turbulence was studied in Ref. [8], where the model was developed which demonstrated that the initial avalanche-like multiplication of vortices leading to turbulence occurs when $q \lesssim 1$ in agreement with experiment [6]. The existence of two regimes in the initial development of vorticity is also supported by earlier simulations by Schwarz who noted that when α (or q) is decreased the crossover from a regime of isolated phase slips to a phase-slip cascades and then to the fully developed vortex turbulence occurs [9]. One can expect that the well developed turbulence occurs when $q \ll 1$, and here we shall discuss this extreme limit. In ^{3}He-B the condition $q \ll 1$ can be realized at temperatures well below $0.6T_c$ [6]. However, we do not consider a very low T where instead of the mutual friction the other mechanisms of dissipation take place such as excitation of Kelvin waves [10] and vortex reconnection [11]. The latter leads to formation of cusps and kinks on the vortex filaments whose fast dynamics creates the burst of different types of excitations in quantum liquids: phonons, rotons, Kelvin waves and fermionic quasiparticles. The burst of gravitational waves from cusps and kinks of cosmic strings was theoretically investigated by the cosmological community (see e.g. [12]), and the obtained results are very important for the superfluid turbulence at a very low temperature.

(5) We expect that even at $q \ll 1$ two different states of turbulence are possible, with the crossover (or transition) between them being determined by q and by another dimensionless parameter $\mathrm{Re}_s = UR/\kappa$, where κ is the circulation around the quantum vortex. The coarse-grained hydrodynamic equation (4) is in fact valid only in the limit $\mathrm{Re}_s \gg 1$, since the latter means that the characteristic circulation of the velocity $\Gamma = UR$ of the large-scale flow substantially exceeds the circulation quantum κ, and thus there are many vortices in the turbulent

flow. When Re_s decreases the quantum nature of vortices becomes more pronounced, and we proceed from the type of the classical turbulence which is probably described by the Komogorov cascade, to the quantum regime which is probably described by the Vinen equations for the average vortex dynamics [13].

Let us consider the possibility of the Kolmogorov state of the superfluid turbulence.

3. Kolmogorov cascade. In classical turbulence, the large Reynolds number $\mathrm{Re} = UR/\nu \gg 1$ leads to the well separated length scales or wave numbers. As a result the Kolmogorov–Richardson cascade takes place in which the energy flows from small wave numbers $k_{min} \sim 1/R$ (large rings of size R of the container) to high wave number $k_0 = 1/r_0$ where the dissipation occurs. In the same manner in our case of the superfluid turbulence the necessary condition for the Kolmogorov cascade is the big ratio of the inertial and dissipative terms in Eq.(4), i.e. $1/q \gg 1$.

In the Kolmogorov-Richardson cascade, at arbitrary length scale r the energy transfer rate to the smaller scale, say $r/2$, is $\epsilon = E_r/t_r$, where $E_r = v_r^2$ is the kinetic energy at this scale, and $t_r = r/v_r$ is the characteristic time. The energy transfer from scale to scale must be the same for all scales, as a result one has

$$\epsilon = \frac{E_r}{t_r} = \frac{v_r^3}{r} = \text{constant} = \frac{U^3}{R} . \tag{8}$$

From this equation it follows that

$$v_r = \epsilon^{1/3} \, r^{1/3} . \tag{9}$$

This must be valid both in classical and superfluid liquids [14]. What is different is the parameter ϵ: it is determined by the dissipation mechanism which is different in two liquids.

From Eq.(7) with $\omega_r = v_r/r$ it follows that as in the classical turbulence the main dissipation occurs at the smallest possible scales, but the structure of ϵ is now different. Instead of $\epsilon = \nu v_{r0}/r_0^2$ in classical liquids, we have now

$$\epsilon \sim q\omega_{r0}U^2 \sim qU^2\frac{v_{r0}}{r_0} = qU^2\epsilon^{1/3} \, r_0^{-2/3} . \tag{10}$$

Since $\epsilon = U^3/R$ one obtains from Eq.(10) that the scale r_0 at which the main dissipation occurs and the characteristic velocity v_{r_0} at this scale are

$$r_0 \sim q^{3/2}R \ , \ \ v_{r0} \sim q^{1/2}U . \tag{11}$$

This consideration is valid when $r_0 \ll R$ and $v_{r0} \ll U$, which means that $1/q \gg 1$ is the condition for the Kolmogorov cascade. In classical liquids the corresponding condition for the well developed turbulence is $\mathrm{Re} \gg 1$. In both cases these conditions ensure that the kinetic terms in the hydrodynamic equations are much larger than the dissipative terms. In the same manner as in classical liquids the condition for the stability of the turbulent flow is $\mathrm{Re} > 1$, one may suggest that the condition for the stability of the discussed turbulent flow is $1/q > 1$. This is supported by observations in ^{3}He-B where it was demonstrated that at high velocity U but at $q > 1$ the turbulence is not developed even after vortices were introduced into the flow [6].

As in the Kolmogorov cascade for the classical liquid, in the Kolmogorov cascade of superfluid turbulence the dissipation is concentrated at small scales,

$$\epsilon \sim qU^2 \int_{r_0}^R \frac{dr}{r}\frac{v_r}{r} \sim qU^2\frac{v_{r0}}{r_0} , \tag{12}$$

while the kinetic energy is concentrated at large scale of container size:

$$E = \int_{r_0}^R \frac{dr}{r}v_r^2 = \int_{r_0}^R \frac{dr}{r}(\epsilon r)^{2/3} = (\epsilon R)^{2/3} = U^2 . \tag{13}$$

The dispersion of the turbulent energy in the momentum space is the same as in classical liquid

$$E = \int_{r_0}^R \frac{dr}{r}(\epsilon r)^{2/3} = \int_{k_0}^{1/R} \frac{dk}{k}\frac{\epsilon^{2/3}}{k^{2/3}} = \int_{k_0}^{1/R} dk E(k),$$
$$E(k) = \epsilon^{2/3}k^{-5/3}. \tag{14}$$

As distinct from the classical liquid where k_0 is determined by viscosity, in the superfluid turbulence the cutoff k_0 is determined by mutual friction parameter q: $k_0 = 1/r_0 = R^{-1}q^{-3/2}$.

4. Crossover to Vinen quantum turbulence. At a very small q the quantization of circulation becomes important. The condition of the above consideration is that the relevant circulation can be considered as continuous, i.e. the circulation at the scale r_0 is larger than the circulation quantum: $v_{r0}r_0 > \kappa$. This gives

$$v_{r0}r_0 = q^2UR = q^2\kappa\mathrm{Re}_s > \kappa, \quad \mathrm{Re}_s = \frac{UR}{\kappa}, \tag{15}$$

i.e. the constraint for the application of the Kolmogorov cascade is

$$\mathrm{Re}_s > \frac{1}{q^2} \gg 1. \tag{16}$$

Another requirement is that the characteristic scale r_0 must be much larger that the intervortex distance l. The latter is obtained from the vortex density in the Kolmogorov state $n_{\mathrm{K}} = l^{-2} = \omega_{r0}/\kappa = v_{r0}/(r_0\kappa)$. The condition $l \ll r_0$ leads again to the equation $v_{r0} r_0 > \kappa$ and thus to the criterion (16).

Note that here for the first time the 'superfluid Reynolds number' Re_s appeared, which contains the circulation quantum. Thus the superfluid Reynolds number is responsible for the crossover or transition from the classical superfluid turbulence, where the quantized vortices are locally aligned (polarized), and thus the quantization is not important, to the quantum turbulence developed by Vinen.

We can now consider the approach to the crossover from the quantum regime – the Vinen state which probably occurs when $\mathrm{Re}_s q^2 < 1$. According to Vinen [13] the characteristic length scale, the distance between the vortices or the size of the characteristic vortex loops, is determined by the circulation quantum and the counterflow velocity, $l = \lambda\kappa/U$, where λ is the dimensionless intrinsic parameter, which probably contains α' and α. The vortex density in the Vinen state is

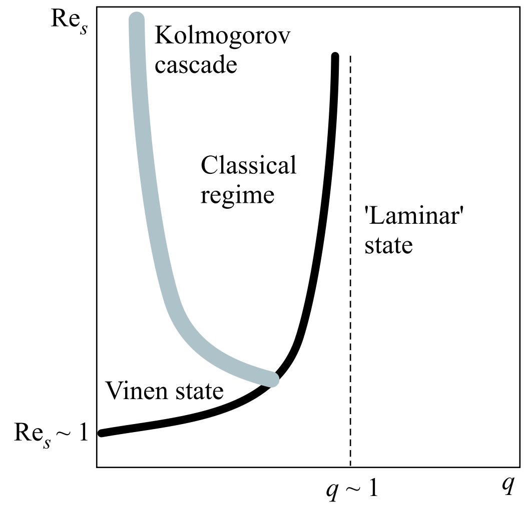
$$n_{\mathrm{V}} = l^{-2} \sim \lambda^2 \frac{U^2}{\kappa^2} = \frac{\lambda^2}{R^2}\mathrm{Re}_s^2. \tag{17}$$

It differs from the vortex density in the Kolmogorov state

$$n_{\mathrm{K}} = \frac{v_{r0}}{\kappa r_0} \sim \frac{U}{q\kappa R} = \frac{1}{R^2}\frac{\mathrm{Re}_s}{q}, \tag{18}$$

which depends not only on the counterflow velocity U, but also on the container size R.

If the crossover between the classical and quantum regimes of the turbulent states occurs at $\mathrm{Re}_s q^2 = 1$, the two equations (17) and (18) must match each other in the crossover region. But this occurs only if $\lambda^2 = q$. If $\lambda^2 \neq q$ there is a mismatch, and one may expect that either the two states are separated by the first-order phase transition, or there is an intermediate region where the superfluid turbulence is described by two different microscopic scales such as r_0 and l. Based on the above consideration one may suggest the following phase diagram of different regimes of collective vortex dynamics in Figure.



Possible phase diagram of dynamical vortex states in Re_s, q plane. At large flow velocity $\mathrm{Re}_s \gg 1$ the boundary between the turbulent and 'laminar' vortex flow approaches the vertical axis $q = q_0 \sim 1$ as suggested by experiment [6]. The thick line separates the developed turbulence of the classical type, which is characterized by the Kolmogorov cascade at small q, and the quantum turbulence of the Vinen type

5. Discussion. The superfluid turbulence is the collective many-vortex phenomenon which can exist in different states. Each of the vortex states can be characterized by its own correlation functions. For example, the states can be characterized by the loop function

$$g(C) = \left\langle \exp\left\{ i(2\pi/\kappa) \oint_C \mathbf{v}\cdot d\mathbf{r} \right\} \right\rangle. \tag{19}$$

In the limit when the length L of the loop C is much larger than the intervortex distance l one may expect the general behavior $g(L) \sim \exp(-(L/l)^\gamma)$ where the exponent γ is different for different vortex states.

One can expect the phase transitions between different states of of collective vortex dynamics. One of such transitions which appeared to be rather sharp has been observed between the 'laminar' and 'turbulent' dynamics of vortices in superfluid ^{3}He-B [6]. It was found that such transition was regulated by intrinsic velocity-independent dimensionless parameter $q = \alpha/(1-\alpha')$. However, it is not excluded that both dimensionless parameters α and α' are important, and also it is possible that only the initial stage of the formation of the turbulent state is governed by these parameters [8]. Another transition (or maybe crossover) is suggested here between the quantum and classical regimes of the developed superfluid turbulence, though there are arguments that the classical regime can never be reached because the vortex stretching is missing in the superfluid turbulence [15].

In principle the parameters α and α' may depend on the type of the dynamical state, since they are obtained by averaging of the forces acting on individual vortices. The renormalization of these parameters $\alpha(L)$ and $\alpha'(L)$ when the length scale L is increasing may also play an important role in the identification of the turbulent states, as in the case of the renormalization-group flow of similar parameters in the quantum Hall effect (see e.g. [16]).

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