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Modulation instability of stokes wave → freak wave

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Formation of wave of large amplitude (freak wave, killer wave) at the surface of ocean is studied numerically. We have observed that freak wave has the same ratio of the wave height to the wave length as for the limiting Stokes wave. When freak wave reaches this limiting state, it breaks. Physical mechanism of freak wave formation is discussed.

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1. Introduction. Waves of extremely large size, alternatively called freak, rogue or giant waves are a well-documented hazard for mariners (see, for instance [1 – 4]). These waves are responsible for loss of many ships and many human lives. No doubts that freak waves are essentially nonlinear objects. They are very steep. In the last stage of their evolution, the steepness becomes infinite, forming a "wall of water". Before this moment, the steepness is higher than for the limiting Stokes wave. Moreover, a typical freak wave is a single event (see [5]). Before breaking it has a crest, three-four (or even more) times higher than the crests of neighbor waves. The freak wave is preceded by a deep trough or "hole in the sea". A characteristic life time of a freak wave is short – ten of wave periods or so. If the wave period is fifteen seconds, this is just few minutes. Freak wave appears almost instantly from a relatively calm sea. Sure, these peculiar features of freak waves cannot be explained by a linear theory. Focusing of ocean waves creates only preconditions for formation of freak waves, which is a strongly nonlinear effect.

It is natural to associate appearance of freak waves with the modulation instability of Stokes waves. This instability is usually called after Benjamin and Feir, however, it was first discovered by Lighthill in [6]. The theory of instability was developed independently in [7] and in [8]. Feir (see [9]) was the first who observed the instability experimentally in 1967.

Slowly modulated weakly nonlinear Stokes wave is described by nonlinear Shrödinger equation (NLSE), derived in [10]. This equation is integrable (see [11]) and is just the first term in the hierarchy of envelope equations describing packets of surface gravity waves. The second term in this hierarchy was calculated by Dysthe in [12], the next one was found a few years ago in [13]. The Dysthe equation was solved numerically by Ablovitz and his collaborates (see [14]).

Since the first work of [1], many authors tried to explain the freak wave formation in terms of NLSE and its generalizations, like Dysthe equation. A vast scientific literature is devoted to this subject. The list presented below is long but incomplete: [13 – 23]. Survey of different possible mechanisms of the freak waves formations is given in [24, 25].

One cannot deny some advantages achieved by the use of the envelope equations. Results of many authors agree in one important point: nonlinear development of modulation instability leads to concentration of wave energy in a small spatial region. This is a "hint" about possible formation of freak wave. On the other hand, it is clear that the freak wave phenomenon cannot be explained in terms of envelope equations. Indeed, NLSE and its generalizations are derived by expansion in series on powers of parameter $\lambda \simeq (Lk)^{-1}$, where k is a wave number, L is a length of modulation. For real freak wave $\lambda \sim 1$ and any "slow modulation expansion" fails. However, the analysis of the NLS-type equations gives some valuable information about formation of freak waves.

Modulation instability leads to decomposition of initially homogeneous Stokes wave into a system of envelope quasi-solitons [26, 27]. This state can be called "quasisolitonic turbulence". In this model solitons can merge, increasing spatial intermittency and leading to establishing of chaotic intense modulations of energy density. So far this model cannot explain formation of freak waves with $\lambda \sim 1$.

Freak wave phenomenon could be explained if the envelope solutions of a certain critical amplitude are unstable, and can collapse. While in the framework of 1D focusing NLSE solitons are stable, the improved model

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must have some threshold in amplitude for soliton stability. Instability of a soliton of large amplitude and further collapse could be a proper theoretical explanation of the freak wave origin.

This scenario was observed in numerical experiment on the heuristic one-dimensional Maida-McLaughlin Tabak (MMT) model (see [28]) of one-dimensional wave turbulence [27]. In the experiments described in the cited paper instability of a moderate amplitude monochromatic wave leads first to creation of a chain of solitons, which merge due to inelastic interaction into one soliton of a large amplitude. This soliton sucks energy from neighbor waves, becomes unstable and collapse up to $\lambda \sim 1$, producing the freak wave.

In our experiments different scenario is observed. Namely, freak wave arppears inside of slightly modulated wave train. Freak wave looks like development of some defect on the periodic grid, which is Stokes wave train.

The most direct way to prove validity of the outlined above scenario for freak wave formation is a straight numerical solution of Euler equation, describing potential oscillations of ideal fluid with a free surface in a gravitational field. This solution can be made by the method published in several articles [29 – 31]. This method is applicable in $2+1$ geometry; it includes conformal mapping of fluid bounded by the surface to the lower half-plane together with "optimal" choice of variables, which guarantees well-posedness of the equations [32].

In the present article we perform experiments for wave trains of steepness $\mu \simeq 0.15$. This experiment could be considered as a simulation of a realistic situation. If a typical steepness of the swell is $\mu \simeq 0.06 \div 0.07$, in caustic area it could easily be two-three times more. In the experiments we start with the Stokes waver train, perturbed by a long wave with twenty time less amplitude. We observe development of modulation instability and finally, the explosive formation of the freak wave that is pretty similar to waves observed in nature.

2. Basic Equations. Suppose that incompressible fluid covers two-dimensional domain

$$-\infty < y < \eta(x,t), \tag{1}$$

where $\eta(x,t)$ – is the shape of surface. The flow is potential, hence

$$V = \nabla\phi, \qquad \nabla V = 0, \qquad \nabla^2\phi = 0. \tag{2}$$

Let $\psi = \phi|_{y=\eta}$ be the potential at the surface and $H = \mathcal{T} + U$ be the total energy. The terms

$$\mathcal{T} = -\frac{1}{2}\int_{-\infty}^{\infty} \psi\phi_n \mathrm{d}x, \quad U = \frac{g}{2}\int_{-\infty}^{\infty} \eta^2(x,t)\mathrm{d}x, \tag{3}$$

are correspondingly kinetic and potential parts of the energy, g is a gravity acceleration and ϕ_n is a normal velocity at the surface. The variables ψ and η are canonically conjugated; in these variables Euler equation of hydrodynamics reads

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}, \qquad \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}. \tag{4}$$

One can perform the conformal transformation to map the domain that is filled with fluid,

$$-\infty < x < \infty, \quad -\infty < y < \eta(x,t), \quad Z = x + iy$$

in Z-plane to the lower half-plane

$$-\infty < u < -\infty, \qquad -\infty < v < 0, \qquad w = u + iv$$

in w-plane. After conformal mapping it is convenient to introduce along with the conformal mapping $Z(w,t)$ complex velocity potential $\Phi(w,t)$. Next, in the paper [33] equations (4) were transformed to a simple form, which is convenient both for numerical simulation and analytical study. Namely, by introducing of new variables

$$R = \frac{1}{Z_w}, \qquad V = i\Phi_z = i\frac{\Phi_w}{Z_w} \tag{5}$$

one can transform system (4) into the following one

$$\begin{aligned} R_t &= i(UR_w - RU_w), \\ V_t &= i(UV_w - RB_w) + g(R-1). \end{aligned} \tag{6}$$

Now complex transport velocity U and B

$$U = \hat{P}(V\bar{R} + \bar{V}R), \quad B = \hat{P}(V\bar{V}). \tag{7}$$

In (7) $\hat{P}$ is the projector operator generating a function that is analytical in a lower half-plane. Here we have omitted all the details which can be found in [29, 33].

3. Numerical approach. Many numerical schemes were developed for the solution of Euler equations describing a potential flow of free-surface fluid in a gravity field. Most of them use the integral equations, which solve the boundary-value problem for Laplace equation [34 – 36]. A survey of the method can be found in [37].

In this article we study modulation instability of Stokes wave. As initial condition we use stationary nonlinear wave train, slightly modulated. This train is unstable with respect to growing of long-scale modulation. This remarkable fact was first established in [6], who calculated a growth-rate of instability in the limit of long-wave perturbation. As far as Lighthill's growth-rate coefficient was proportional to the wave number of perturbation length, the result was in principle incomplete: somewhere at short scales the instability must be arrested. The complete form of the growth-rate coefficient was found independently in [7, 8, 10].

We apply the spectral code to solve equations (6). We should mention that conformal mapping is a routine approach to study the stationary Stokes wave. The equations for their Fourier coefficients were solved numerically by many authors (see, for instance [38]). The idea to implement conformal mapping for simulation of essentially nonstationary wave dynamics emerged in the beginning of eighties (see [39]). As far as equations (6) were not derived at that time, the authors have been using the quasi-Lagrangian approach to fluid dynamics. After some experiments and discussion of their results, the idea to use the conformal mapping war abundant for the following reason: conformal mapping is not good for resolution of wedge-type singularities, naturally appearing on the free surface of fluid. This reason is serious if the spatial mesh is sparse. However, modern computers make possible to use very fine meshes consisting of more than million points or spectral modes. Thus, this argument is not tenant any more.

Our recent experiments are sufficiently accurate: we use 10^5 to $2 \cdot 10^6$ harmonics. We solve equations (6) in the periodic domain $0 < x < 2\pi$, putting $g = 1$. The initial data are chosen as a combination of the exact Stokes wave (wave number $k = 10$, steepness $ka = \mu = 0.15$) and a long monochromatic wave with wave number $k = 1$ and a moderate amplitude $5 \cdot 10^{-2}$. This relatively high level of perturbation is chosen deliberately to make shorter the period of exponential instability growth that is not interesting for us. At given conditions, the maximum growth-rate is

$$\gamma_{\max} \simeq \frac{\sqrt{10}}{2} \cdot 0.15^2 \simeq 0.035$$

and $\gamma_{\max}^{-1} \simeq 28.6$. The period of initial wave $T_0 = 2\pi/\sqrt{10}$. The simulation is continued until $T \simeq 458.842$, that is more than sixteen inverse growth-rates. We performed the computations with double precision, with the number of modes doubled as far as amplitude of the last mode reached 10^{-15}. The maximum number of modes was two millions.

We observed a short period of exponential growth of perturbation, then, some intermediate regime of intensive modulation, which ends up with explosive formation of one single freak wave. Pictures of surface shape before breaking at times $T = 442$ and $T = 458.56$ are presented on Fig.1 and Fig.2.

The time interval $T = 442$ and $T = 458.56$ contains seven periods of initial wave only. One can see fast,

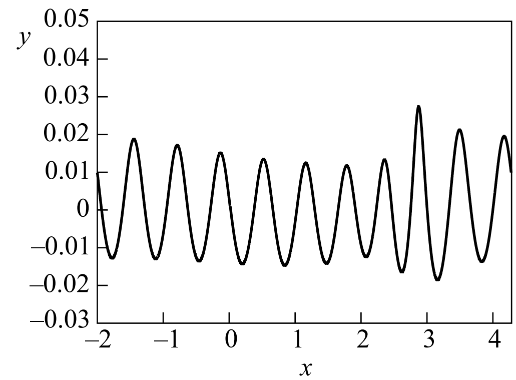

Fig.1. The shape of surface at $T = 422$

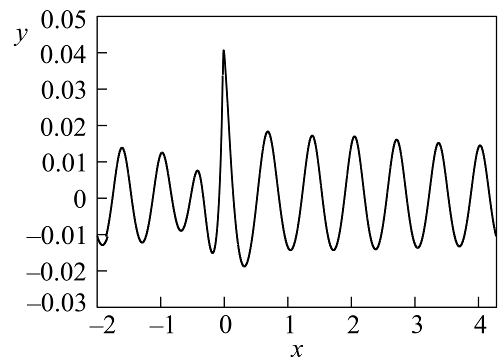

Fig.2. The shape of surface at $T = 458.56$

non-monotonic formation of the freak wave. At this moment the freak wave is more steep than the Stokes wave of limiting amplitude. Amplitudes of waves, preceding the freak wave, are relatively small (three times less). One can see a trough just ahead of freak wave. This is so-called "hole in the water" (marine folklore) which precedes freak wave. Fig.3 demonstrates the fine structure of surface shape near the wave crest.

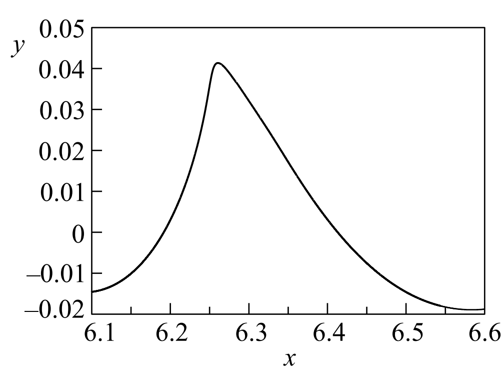

Fig.3. The shape of surface near the wave crest at $T = 458.61$

We managed to continue our simulation until the moment $T = 458.842$. The zoomed shape of the surface at that time is presented on Fig.4.

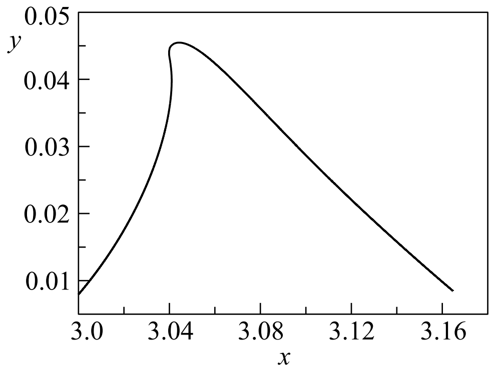

Fig.4. The shape of surface near the wave crest at $T = 458.842$

One can see that near the crest the front face of the wave is very steep. This is really "wall of water". In some region the steepness is even negative. The curvature of the shape is plotted on Fig.5.

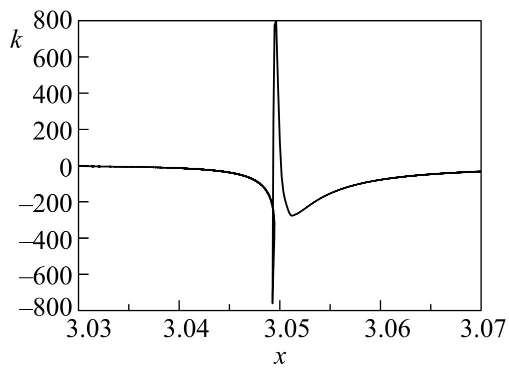

Fig.5. Curvature (k) of the surface at $T = 458.842$

This is actually a breaking wave. Moreover, in all our experiments at the moment of breaking we have observed that the ratio of wave height to the wavelength is practically the same, and close to that of limiting Stokes wave, 0.141.

Note, that the maximum value of the freak wave height is three times higher than the height of the initial wave. Growing of wave hight up to this level from the level of significant wave height takes less than ten wave periods. This is a really fast process; it is three times faster than the developing of modulation instability.

Fig.6 display the evolution of spatial density of kinetic energy (in the domain [5.5–9.5]), where the breaking takes place.

One can see that this evolution is non-monotonous. The density oscillates in time and finally condensates

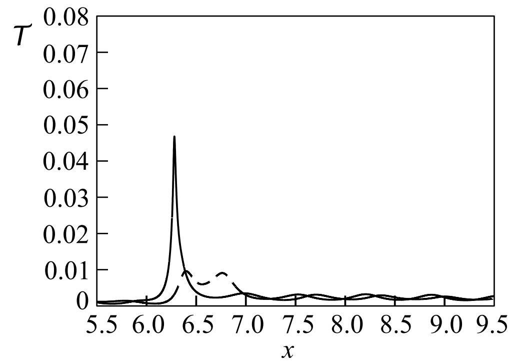

Fig.6. The density of kinetic energy just before breaking at $T = 456$ (dashed line), and at the moment of breaking at $T = 458.5$ (solid line)

in one very narrow wave crest. In general, the whole process of freak wave formation is non-monotonous. We can say that the freak wave "runs" over wave crests until one of them reaches extremely high amplitude. This behavior can be easily explained by difference of phase and group velocities: the energy propagates with group velocity that is twice less than the phase velocity. Fig.7

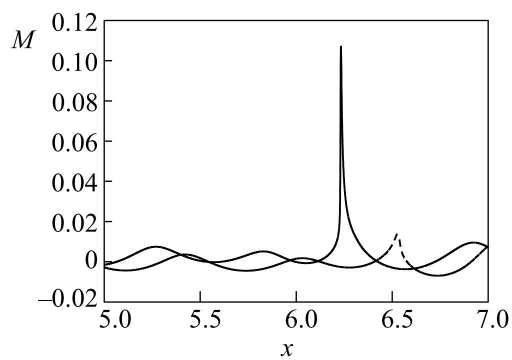

Fig.7. Distribution of momentum (M) before (dashed line) and after (solid line) breaking

demonstrates distribution of horizontal momentum before and after breaking, at $T = 455$ and $T = 456$. One can see that the process of momentum concentration in a moving but localized area is monotonous. Definitely, this behavior can be explained by the fact that momentum is a conservative quantity.

4. Conclusions. Let us summarize our numerical experiments. Certainly, they reproduce the most apparent features of freak waves: single wave crests of very high amplitude, exceeding the significant wave height more than three times, appearing from "nowhere" and reaching full height in a very short time, less than ten periods of surrounding waves. The singular freak wave is proceeded by the area of diminished wave amplitudes. Final "fate" of the freak wave is breaking. The ratio of freak wave height to its wavelength is practically the same, being close to the limiting Stokes wave, 0.141. Freak wave moves with the group velocity.

In our experiments, the freak wave appears as a result of development of modulation instability (if the threshold of the instability is not exceeded no freak waves appear at all). Than it takes a long time for the onset of instability to create a freak wave. Meanwhile, the freak wave appears only after fifteenth inverse growth-rates of instability. What happens after developing of instability but before formation of freak wave? This stage could be considered as a development of some defect on the periodic grid. This grid is just initial Stokes wave train. Similar picture was observed in [40] where breaking of wave in the group was studied.

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1. R. Smith, J. Fluid Mech. **77**, 417 (1976).
2. R. G. Dean, in: *Water Wave Kinetics*, Eds. A. Torum and O. T. Gudmestad, Kluwer, 1990, p. 609.
3. I. V. Lavrenov, Natural Hazards **17**, 117 (1998).
4. G. A. Chase, Big wave, (2003); http://bell.mma.edu/ achase/NS-221-Big-Wave.html.
5. V. D. Divinsky, B. V. Levin, L. I. Lopatikhin et al., Doklady Earth Science **395**, 438 (2004).
6. M. J. Lighthill, J. Inst. Math. Appl. **1**, 269 (1965).
7. T. B. Benjamin and J. E. Feir, J. Fluid Mech. **27**, 417 (1967).
8. V. E. Zakharov, J. Teor. Prikl. Fiz. **51**, 668 (1966) (in Russian); Sov. Phys. JETP **24**, 455 (1967).
9. J. E. Feir, Proc. R. Soc. Lond. **A299**, 54 (1967).
10. V. E. Zakharov, J. Appl. Mech. Tech. Phys. **9**, 190 (1968).
11. V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972).
12. K. B. Dysthe, Proc. Roy. **A369**, 105 (1979).
13. K. Trulsen and K. B. Dysthe, Wave Motion **24**, 281 (1996).
14. M. I. Ablovitz, D. Hammack, J. Henderson, and C. M. Scholder, Phys. Rev. Lett. **84**, 887 (2000); Physica **D152-153**, 416 (2001).
15. M. Onorato, A. R. Osborne, M. Serio, and T. Damiani, in: *Rogue waves 2000: Brest, France, November 2000*, Eds. M. Olagnon and G. A. Athanassoulis, Ifremer, 2001, p. 181.
16. M. Onorato, A. R. Osborne, and M. Serio, Phys. Lett. **A275**, 386 (2000).
17. M. Onorato, A. R. Osborne, M. Serio, and S. Bertone, Phys. Rev. Lett. **86**, 5831 (2001).
18. M. Onorato, A. R. Osborne, and M. Serio, Phys. of Fluids **14**, L25 (2002).
19. D. H. Peregrine, J. Austral. Math. Soc. **B25**, 16 (1983).
20. D. H. Peregrine, D. Skyner, M. Stiassnie, and J. W. Dold, in: *Proc. 21th Intl. Conf. on Coastal Engng.*, Vol. **1**, Chap. 54, 1988, p. 732.
21. K. Trulsen and K. B. Dysthe, in: *Proc. 21st Symposium on Naval Hydrodynamics*, 1997, p. 550; http://www.nap.edu/books/0309058791/html/550.html
22. K. Trulsen, in *Rogue waves 2000: Brest, France*, November, 2000, Eds. M. Olagnon and G. A. Athanassoulis, Ifremer, 2001, p. 265.
23. K. Trulsen, I. Kliakhandler, K. B. Dysthe, and M. G. Velarde, Phys. Fluids **24**, 32 (2000); D. Clamond and J. Grue, C.R.Mecanique **330**, 575 (2002).
24. C. Kharif and E. Pelinovsky, Europ. J. Mech. B/Fluids, **22**, 603 (2003).
25. A. A. Kurkin and E. N. Pelinovsky, *"Freak waves: facts, theory and modeling"*, Nizhnii Novgorod University, 2004.
26. V. E. Zakharov and E. A. Kuznetsov, JETP **86**, 1035 (1998).
27. V. E. Zakharov, F. Dias, and A. N. Pushkarev, Phys. Reports **398**, 1 (2004).
28. A. Majda, D. McLaughlin, and E. Tabak, J. Nonlinear Science **7**, 9 (1997).
29. A. I. Dyachenko, E. A. Kuznetsov, M. D. Spector, and V. E. Zakharov, Phys. Lett. **A221**, 73 (1996).
30. V. E. Zakharov, A. I. Dyachenko, and O. A. Vasilyev, Eur. J. of Mech. B-Fluids **21**, 283 (2002).
31. V. E. Zakharov, Amer. Math. Soc. **(Ser.2) 182**, 167 (1998).
32. A. I. Dyachenko, in: *Proc. of the II Intern. conf. "Frontiers of Nonlinear Physics"*, Nizhny Novgorod-St.Petersburg, Russia, July 5-12, 2004.
33. A. I. Dyachenko, Doklady Mathematics **63**, 115 (2001).
34. J. W. Dold and D. H. Peregrine, in: *Proc. 20th Intl Conf. on Coastal Engng.*, Vol. **1**, Chap. 13, 1986, p. 163.
35. D. G. Dommermuth and D. K. P. Yue, J. Fluid Mech. **184**, 267 (1987).
36. J. W. Dold, J. Computat. Phys. **103**, 90 (1992).
37. W. Tsai and D. Yue, Annu. Rev. Fluid Mech. **28**, 249 (1996).
38. J. A. Zufiria and P. G. Saffman, Stud. Appl. Maths. **74**, 259 (1986).
39. D. Meison, S. Orzag, and M. Izraely, J. Computational Physics, **40**, 345 (1981).
40. J. Song and M. L. Banner, J. Phys. Oceanogr. **32**, 2541 (2002).