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Non-Fermi liquid theory of the quantum Hall effects

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Within the Grassmannian $U(2N)/U(N)\times U(N)$ non-linear σ model representation of localization one can study the low energy dynamics of both the free and interacting electron gas. We study the cross-over between these two fundamentally different physical problems. We show how the topological arguments for the exact quantization of the Hall conductance are extended to include the Coulomb interaction problem. We discuss dynamical scaling and make contact with the theory of variable range hopping.

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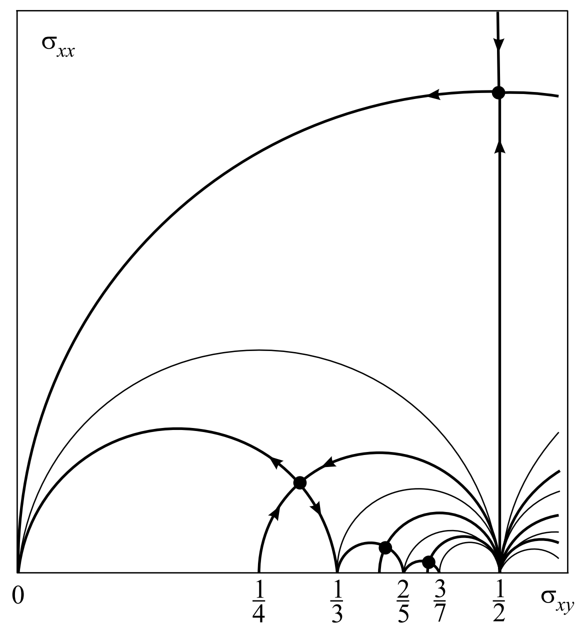
Over the last few years, much effort has been devoted to the problem of localization and interaction effects in the quantum Hall regime [1 – 6]. By now it is well understood that the Coulomb interaction problem falls in a non-Fermi liquid universality class of transport problems with a novel symmetry, named $\mathcal{F}$ *invariance* [2]. Although the results for scaling are in many ways similar to those obtained for the free electron gas [7], it is important to bear in mind that the Coulomb interaction problem is a far richer one. Unlike the free particle problem, for example, the field theory for interacting particles provides the platform for a unification of the *fractional quantum Hall regime* and the quantum theory of *metals* [2, 4 – 6]. The principal features of this unification are encapsulated in a scaling diagram for the *longitudinal* and *Hall conductances* σ_{xx} and σ_{xy} respectively (Figure). The Finkelstein approach to localization and interaction phenomena [8, 9], the topological concept of an instanton vacuum [10] as well as the Chern Simons statistical gauge fields [11] are all essential in composing this diagram.



Unified scaling diagram for the quantum Hall effects in the σ_{xx}, σ_{xy} conductivity plane. The arrows indicate the direction towards the infrared

The main objective of this Letter is to embark on the most fundamental aspect of the theory, the *observability* and *precision* of the quantum Hall effect. This experimental phenomenon is represented in Figure by the infrared stable fixed points located at $\sigma_{xx}=0$ and $\sigma_{xy}=k$ (integer effect) as well as $\sigma_{xy}=k/(2k+1)$ (Jain series) [12]. These fixed points, however, define the strong coupling phase of the unifying action where analytic work is generally impossible. In spite of ample experimental evidence for its existence, the *robust quantization* of the Hall conductance has yet to be established as a fundamental but previously unrecognized feature of the topological θ vacuum concept [10].

In what follows we shall benefit from the advancements reported in Ref. [13]. In particular, since the Finkelstein theory is formally defined as a σ model on the Grassmann manifold $U(2N)/U(N)\times U(N)$ with N equal to N_r (number of *replica's*) times N_m (number of *Matsubara frequencies*), we can now use our general knowledge on the strong coupling behavior of the theory

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and probe, for the first time, the quantum Hall phases in the interacting electron gas.

To achieve our goals we first shall outline some of the recent advancements in the field. It is important to emphasize that the *complete* effective action for interacting particles now exists [2]. This action includes the coupling to external potentials and/or Chern Simon gauge fields. This leads to a detailed understanding of the electrodynamic $U(1)$ gauge invariance and provides invaluable information on the renormalization of the theory that was not available before.

Secondly, it is necessary to have a more detailed understanding of how the subtleties of interaction effects can be understood as a field theory. For this purpose we report new results on the Grassmannian non-linear σ model with $N_r = 0$ and varying N_m. These show explicitly how the cross-over takes place between a theory of *free particles* at finite values of N_m and a *many body* theory that is generally obtained in the limit $N_m \to \infty$ only. Armed with these insights we next point out how the Coulomb interaction problem, at zero temperature (T), displays the general topological features and θ dependence that were discovered in Ref. [13].

As a third and final step toward the strong coupling phase we discuss the subject of dynamical scaling. As a unique product of our effective action procedure we obtain a distinctly different behavior at finite T, depending on the specific regime of the interacting electron gas that one is interested in. We establish, at the same time, the contact with the theory for *variable range hopping* [14].

• *The action.* Following Finkelstein [8], the effective quantum theory for disordered (spin polarized or spinless) electrons is given in terms of a generalized σ model involving the unitary matrix field variables $Q^{\alpha\beta}_{nm}(\mathbf{r})$ which obey $Q^2(\mathbf{r}) = \mathbf{1}$. Here, α, β represent the replica indices, n, m are the indices of the Matsubara frequencies $\omega_k = \pi T(2k+1)$. In terms of ordinary unitary rotations $\mathcal{T}^{\alpha\beta}_{nm}$ one can write

$$Q = \mathcal{T}^{-1}\Lambda\mathcal{T}, \qquad \Lambda = \Lambda^{\alpha\beta}_{nm} = \mathrm{sign}\,(\omega_n)\mathbf{1}^{\alpha\beta}_{nm} \qquad (1)$$

indicating that the Q describes a Goldstone manifold of a broken symmetry between positive and negative frequencies. A $U(1)$ gauge transformation in frequency space is represented by a unitary matrix $\mathcal{W}^{\alpha\beta}_{nm}$

$$\mathcal{W} = \exp\{i\sum_{n,\alpha}\phi^\alpha(\mathbf{r},\nu_n)I^\alpha_n\}, \qquad (2)$$

with $\nu_n = 2\pi Tn$. Here, $[I^\gamma_k]^{\alpha\beta}_{nm} = \delta^{\alpha\gamma}\delta^{\beta\gamma}\delta_{n,m+k}$ denote the $U(1)$ generators. In finite frequency space with a cut-off (N_m), the I matrices no longer span a $U(1)$ algebra. To define the $U(1)$ gauge invariance in a truncated frequency space we have developed a set of rules ($\mathcal{F}$ *algebra* [2]). These involve one more (frequency) matrix, $\eta^{\alpha\beta}_{nm} = n\delta^{\alpha\beta}_{nm}$, that is used to represent ω_n. The effective action for electrons in a static magnetic field B and coupled to external potentials and/or Chern Simons fields $a^\alpha_\mu(\mathbf{r},\nu_n)$ with $\nu_n \neq 0$, can now be written as [2]

$$S_{\text{eff}} = S_\sigma + S_F + S_U + S_0. \qquad (3)$$

Here, S_σ is the free electron piece [7]

$$S_\sigma = -\frac{\sigma_{xx}}{8}\int d\mathbf{r}\,\mathrm{tr}\,[D_i,Q][D_i,Q] + + \frac{\sigma_{xy}}{8}\int d\mathbf{r}\,\mathrm{tr}\,\epsilon_{ij}Q[D_i,Q][D_j,Q], \qquad (4)$$

where $D_j = \nabla_j - i\sum_{n\alpha}a^\alpha_j(\mathbf{r},\nu_n)I^\alpha_n$ is the covariant derivative. Next, the two pieces S_F and S_U are linear in temperature T and represent interaction terms. S_F is gauge invariant and contains the *singlet interaction* term [8]

$$S_F = \pi zT\int d\mathbf{r}\left[\sum_{\alpha n}\mathrm{tr}\,I^\alpha_nQ\,\mathrm{tr}\,I^\alpha_{-n}Q + 4\mathrm{tr}\,\eta Q - 6\mathrm{tr}\,\eta\Lambda\right]. \qquad (5)$$

The (Coulomb) term S_U contains the scalar potential

$$S_U = -\pi T\sum_{\alpha n}\int d\mathbf{r}d\mathbf{r}'\left[\mathrm{tr}\,I^\alpha_nQ(\mathbf{r}) - \frac{1}{\pi T}\tilde{a}^\alpha_0(\mathbf{r},\nu_{-n})\right]\times \times U^{-1}(\mathbf{r}-\mathbf{r}')\left[\mathrm{tr}\,I^\alpha_{-n}Q(\mathbf{r}') - \frac{1}{\pi T}\tilde{a}^\alpha_0(\mathbf{r}',\nu_n)\right]. \qquad (6)$$

The S_0 contains the magnetic field $b^\alpha = \epsilon_{ij}\nabla_i a^\alpha_j$:

$$S_0 = -\frac{\rho_B^2}{2\rho T}\int d\mathbf{r}\sum_{\alpha n}b^\alpha(\mathbf{r},\nu_n)b^\alpha(\mathbf{r},\nu_{-n}). \qquad (7)$$

We have defined (dropping the replica index α on a_μ)

$$\tilde{a}_0 = a_0 - \frac{i\rho_B}{\rho}b, \qquad U(q) = \rho^{-1} + U_0(q). \qquad (8)$$

Here, the density of states $\rho = (\partial n/\partial\mu)_{T,B}$ and the quantity $\rho_B = (\partial n/\partial B)_{T,\mu}$ are thermodynamic quantities, n and μ being the particle density and the chemical potential respectively. The statement of gauge invariance now means that the theory is invariant under the following transformation

$$Q \to \mathcal{W}^{-1}Q\mathcal{W}, \qquad a_\mu \to a_\mu + \partial_\mu\phi. \qquad (9)$$

Using Eq. (9) it is easy to see that the action is invariant under spatially independent gauge transformations $\phi = \phi(\nu_n)$ provided the interaction potential U_0 has an infinite range. This global invariance, termed $\mathcal{F}$ *invariance*, is an exact symmetry of the Coulomb interaction problem which in two spatial dimensions is represented by $U_0^{-1}(q) = \Gamma|q|$.

• *Static versus dynamic response.* Our introduction of external potentials (statistical gauge fields) a_μ can be exploited immediately to elucidate fundamental aspects of the quantum transport problem in strong B. For this purpose we consider $S_{\rm eff}[a_\mu]$ obtained after elimination of the Q fields. Defining the particle density $n_m = T\delta S_{\rm eff}/\delta a_0(\nu_{-m})$ we obtain, at a tree level, the continuity equation [2]

$$\nu_m(n_m + i\sigma_{xy}b) = \nabla\cdot\Big[\sigma_{xx}(\mathbf{e} + \nabla(U_0 n_m)) - - D_{xx}\nabla(n_m + i\rho_B b)\Big]. \qquad (10)$$

Here, $D_{xx} = \sigma_{xx}/\rho$ denotes the diffusion constant and $\mathbf{e}$, b are the external electric and magnetic fields respectively. This result is familiar from the theory of metals [15] where the quantity ρ_B is usually neglected. Notice that in the static limit $\nu_n \to 0$ both quantities σ_{ij} drop out and the equation now contains the thermodynamic quantities ρ, ρ_B and U_0 only. Since the fields $a_\mu(\mathbf{r}, \nu_n = 0)$ are completely decoupled from the Q field variables, the static response is actually determined by a different, underlying theory. This means that ρ, ρ_B, U_0 and hence S_U and S_0 should not have any quantum corrections in general, either perturbatively or nonperturbatively. This observation can be used as a general physical constraint that must be imposed on the quantum theory. The only quantities that are allowed to have quantum corrections are the transport parameters σ_{xx}, σ_{xy}, and the singlet interaction amplitude z.

As an important check on the statements of gauge invariance and renormalization, we have evaluated the quantum theory in $2+\epsilon$ spatial dimensions to order ϵ^2. The results of the computation, along with an extensive analysis of dynamical scaling, have been reported in Ref. [6].

• $\mathcal{F}$ *invariance.* The renormalizability can be addressed more formally, by making contact with the theory of ordinary non-linear sigma models [16]. For this purpose we drop the external potentials from the action and recall that for finite size matrices Q, operators like S_F play the role of infrared regulators that do not affect the singularity structure of the theory at short distances. We know in particular that the theory is renormalizable in two dimensions. Besides the coupling constant or σ_{xx}, one additional renormalization constant is needed for the operators linear in the Q matrix field and two more are generally needed for the operators bilinear in the Q (i.e. the symmetric and anti-symmetric representation respectively) [17]. These general statements apply to the Finkelstein action as well since the latter only demands that the number of Matsubara frequencies N_m is taken to infinity (along with $N_r \to 0$). To completely undust this point we have computed the cross over functions for the theory where the quantity $U^{-1}(\mathbf{r}-\mathbf{r}')$ in S_U, Eq. (6), is replaced by its most relevant part

$$U^{-1}(\mathbf{r}-\mathbf{r}') \to z(1-c)\delta(\mathbf{r}-\mathbf{r}'). \qquad (11)$$

Notice that $0 < c < 1$ represents the finite range interaction case. The extreme cases $c = 0$ and 1 describe the free electron gas and the Coulomb system respectively. $\mathcal{F}$ invariance is retained for $c = 1$ only and broken otherwise.

The following renormalization group functions have been obtained for the parameters z, c and the dimensionless resistance $g = \mu^\epsilon/\pi\sigma_{xx}$ in $2+\epsilon$ spatial dimensions (μ denotes an arbitrary momentum scale) [18]

$$\frac{dg}{d\ln\mu} = \epsilon g - 2g^2\left[f + \frac{1-c}{c}\ln(1-cf)\right], \qquad (12)$$

$$\frac{d\ln z}{d\ln\mu} = gcf, \qquad (13)$$

$$\frac{dc}{d\ln\mu} = gc(1-cf). \qquad (14)$$

Here, $f = M^2/(\mu^2+M^2)$ is a μ-dependent function with $M^2 = 8\pi z T N_m/\sigma_{xx}$ which depends on the cut-off N_m.

For $f = 0$ ($\mu \gg M$ or short distances) we obtain the well known results for free particles [16, 17], i.e. $dg/d\ln\mu$ has no one-loop contribution, z has no quantum corrections in general and the result for c coincides with the renormalization of symmetric operators, bilinear in Q.

For $f = 1$ ($\mu \ll M$ or large distances), we obtain the peculiar Finkelstein results of the interacting electron gas [3, 9]. The symmetry breaking parameter c now affects the renormalization of all the other parameters. The concept of $\mathcal{F}$ invariance ($c = 1$) manifests itself as a new (non-Fermi liquid) fixed point in the theory. The problem with $0 < c < 1$ lies in the domain of attraction of the Fermi liquid line $c = 0$ which is stable in the infrared.

Notice, however, that the $\mathcal{F}$ invariant fixed point $c = 1$ only exists if the mass M in the theory remains finite at zero T. This clearly shows that, in order for $\mathcal{F}$ invariance to represent an exact symmetry of the problem, N_m must be infinite. The time τ plays the role of an extra, non-trivial dimension and this dramatically complicates the problem of plateau transitions in the quantum Hall regime. The Coulomb interaction problem, unlike the free electron theory, is given as a $2+1$ dimensional field theory, thus invalidating any attempt toward exact solutions of the experimentally observed critical indices [19 – 22].

• *The quantum Hall effect.* Next, we turn to the most interesting aspect of the theory, the σ_{xy} term (θ term), which is invisible in perturbative expansions. However, we may proceed along the same lines as pointed out in Ref. [13] and separate, in the theory for $T=0$, the *bulk* quantities from the *edge* quantities [23, 10]

$$\sigma_{xy} = \nu_B = k + \frac{\theta}{2\pi}, \qquad k \in \mathbb{Z}, \qquad -\pi < \theta \leq \pi, \tag{15}$$

where ν_B stands for the filling fraction. Specifying to the Coulomb interaction problem ($c=1$) in two spatial dimensions, we next make use of the principle of $\mathcal{F}$ invariance and formulate an effective action for the edge. Introducing a change of variables $Q = t^{-1}Q_0 t$ [10], we now have [24] ($q = t^{-1}\Lambda t$)

$$\begin{gathered} S_{\text{eff}}[q] = S_{\text{bulk}}[q] + 2\pi i k C[q], \\ e^{S_{\text{bulk}}[q]} = \int_{\partial V} D[Q_0] e^{\tilde{S}_\sigma[t^{-1}Q_0 t] + S_F[t^{-1}Q_0 t]}. \end{gathered} \tag{16}$$

Here $\tilde{S}_\sigma$ is the same as S_σ with σ_{xy} replaced by its *unquantized* bulk piece θ. Recall that the symbol ∂V reminds us that the functional integral is performed with a fixed value $Q_0 = \Lambda$ at the edge. It is important to notice that the interaction piece S_F cannot be left out since it affects, following Eqs. (12)-(14), the renormalization of the theory at $T=0$.

The definition of $S_{\text{eff}}[t]$ is precisely the same as the *background field* methodology adapted to the Coulomb interaction problem [1, 3]. The result is of the form

$$S_{\text{bulk}}[q] = \tilde{S}'_\sigma[q] + S'_F[q], \tag{17}$$

where the primes indicate that the parameters σ_{xx}, θ and z are replaced by renormalized ones, σ'_{xx}, θ' and z' respectively, which are defined for system size L.

This leads to the most important statement of this Letter which says that, provided a *mass* is generated for *bulk* excitations, the renormalized theory $\sigma'_{xx} = \sigma_{xx}(L)$, $\theta' = \theta(L)$ and $z' = z(L)$ should vanish for large enough L, i.e. the bulk of the system is insensitive to changes in the boundary conditions except for corrections exponentially small in L. Under these circumstances $S_{\text{eff}}[q]$ reduces to the action of *massless chiral edge excitations* [4, 5]. The integer k equals the number of edge modes and is now identified as the *quantized Hall conductance*.

These results describe the strong coupling "integer quantum Hall" fixed points (Figure) that were previously conjectured on phenomenological grounds. From the weak coupling side, a detailed analysis of Eqs. (16) and (17) leads to the following expressions [1] for the renormalization group functions

$$\frac{d\sigma_{xx}}{d\ln\mu} = \beta_\sigma = \beta^0_\sigma(\sigma_{xx}) + D\sigma^2_{xx} e^{-2\pi\sigma_{xx}} \cos\theta, \tag{18}$$

$$\frac{d\sigma_{xy}}{d\ln\mu} = \beta_\theta = D\sigma^2_{xx} e^{-2\pi\sigma_{xx}} \sin\theta, \tag{19}$$

$$\frac{d\ln z}{d\ln\mu} = \gamma_z = \gamma^0_z(\sigma_{xx}) + D_\gamma \sigma_{xx} e^{-2\pi\sigma_{xx}} \cos\theta. \tag{20}$$

Here, $D \approx 13.58$ and $D_\gamma \approx 2.26$ are determined by the instanton determinant [24] and β^0_σ and γ^0_z are the perturbative results that recently have been extended to two-loop order ($\mathcal{A} \approx 1.64$) [3, 6]

$$\beta^0_\sigma(\sigma_{xx}) = \frac{2}{\pi} + \frac{4\mathcal{A}}{\sigma_{xx}}, \tag{21}$$

$$\gamma^0_z(\sigma_{xx}) = \frac{1}{\pi\sigma_{xx}} + \frac{18+\pi^2}{6\pi^2\sigma^2_{xx}}. \tag{22}$$

In summary, there is now fundamental support, both from the weak and strong coupling side, for the scaling diagram of the integral quantum Hall effect [1].

• *Finite T.* At finite T the infrared of the system is controlled by the interaction terms S_F and S_U. In this case one must go back to the original theory (Eqs. (3)-(7)) and obtain the transport parameters from linear response in the field a_μ [3]. Specifying to the $a_0 = 0$ gauge as well as $\nabla \cdot \mathbf{a} = \nabla \times \mathbf{a} = 0$ we can write

$$S_{\text{eff}}[a] = T \sum_{n>0} \int d\mathbf{r}\, \nu_n \left[\sigma'_{xx}\delta_{ij} + \sigma'_{xy}\epsilon_{ij}\right] a_i(\nu_n) a_j(-\nu_n), \tag{23}$$

where the expressions for σ'_{ij} are known as the *Kubo formulae* [3]. We stress that these expressions are exactly the same as those obtained from the background field procedure, $\sigma'_{xx} = \sigma_{xx}(L)$ and $\sigma'_{xy} = k + \theta(L)/(2\pi)$ (Eqs. (17) and (18)-(20)), provided $S_{eff}[a]$ is evaluated at $T=0$ and with $Q = \Lambda$ at the edge [2, 3].

The scaling results at finite T generally depend on the specific regime and/or microscopics of the disordered electron gas that one is interested in. Here we consider the most interesting cases where $\theta \approx \pm\pi$ and $\theta \approx 0$ respectively. The first case is realized when the Fermi level passes through the center of the Landau band where the electron gas is quantum critical and the transition takes place between adjacent quantum Hall plateaus [7]. Provided the bare parameter σ_{xx} of the theory is close to the critical fixed point σ^*_{xx} at $\sigma_{xy} = 1/2$ (Figure) the following universal scaling law is observed [7]

$$\sigma'_{xx} = \sigma_{xx}(X), \qquad \sigma'_{xy} = k + \frac{\theta(X)}{2\pi}, \tag{24}$$

where $X = (zT)^{-\kappa}\Delta\nu_B$. Here $\Delta\nu_B$ is the filling fraction ν_B of the Landau levels relative to the critical value ν_B^* which is half-integer. The correlation (localization) length ξ of the electron gas diverges algebraically $\xi \propto |\Delta\nu_B|^{-1/\nu}$. The critical indices κ and ν are a major objective of experimental research [19–22] and the results have been discussed extensively and at many places [1, 4, 6, 14].

Next we consider $\theta \approx 0$ which is entirely different. This happens when the Fermi energy is located at the tail end of the Landau bands corresponding to the center of the quantum Hall plateau. The bare parameter σ_{xx} of the theory is now close to zero [4]. This means that the T dependence is determined by the strong coupling asymptotic of the renormalization (θ, $\sigma_{xx} \to 0$). Notice that the γ_z function (Eqs. (18)–(20) and (21)–(22)) indicates that the singlet interaction term S_F eventually renders irrelevant as compared to the Coulomb term S_U (with $U^{-1}(q) = \Gamma|q|$) which, as we mentioned before, is not affected by the quantum theory. One now expects S_U to become the dominant infrared regulator such that the scaling variable X in Eq. (24) is now given by $X = T\Gamma\xi$.

This asymptotic limit of the theory can be identified as the Effros–Shklovskii regime of *variable range hopping* for which the following result is known $\sigma'_{xx} = \sigma_{xx}(T\Gamma\xi) = \exp(-2/\sqrt{T\Gamma\xi})$ [14]. We therefore conclude that the dynamics of the electron gas is generally described by distinctly different physical processes and controlled by completely different fixed points in the theory.

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