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TOPOLOGY OF VORTEX-SOLITON INTERSECTION: INVARIANTS AND TORUS HOMOTOPY

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We discuss topology relevant to the intersection of nonsingular 4π-vortex lines with a planar transverse soliton in superfluid ^{3}He-A which was recently observed in Helsinki. The essential part of the problem consists in finding a homotopy classification of mappings $S^2 \times S^1 \to S^2$ and $S^1 \times S^1 \times S^1 \to S^2$. This classification is achieved, and an analytical expression for the topological invariant is found, analogous to that for the Hopf invariant $H = (1/4\kappa^2)\int \mathbf{v}_s \,\mathrm{rot}\, \mathbf{v}_s dV$ of mappings $S^3 \to S^2$.

Many different types of topological defects have been suggested in condensed matter with broken symmetry, a number of them having been experimentally discovered by now. A perfect mathematical tool for their investigation is given by homotopy theory that have thus been extensively used in physics [1, 2]. In many important cases it is sufficient just to know the order parameter space and be able to calculate its homotopy groups π_n. In particular, knowledge of homotopy groups permits to classify possible free monopoles, vortices and interfaces, defects that appear ubiquitous in the Universe [2, 3]. Classification of defects caused by some sort of boundary conditions or confined to other defects occurs more laborious, leading in the simplest cases to relative homotopy groups [4, 5].

Recently an experimental evidence for coexistence of planar soliton with piercing it nonsingular 4π-vortices in rotating ^{3}He-A was reported [6]. The purpose of the present article is to extend the discussion of underlying topology, including analytical expressions for topological invariants. We will show that the essential part of the problem consists in topological classification of mappings $S^2 \times S^1 \to S^2$ and $S^1 \times S^1 \times S^1 \to S^2$. The latter set of mappings is sometimes called "torus homotopy group" of 2-sphere $T^3(S^2)$ and alleged to be unknown [7], though actually it was first found by L. Pontrjagin in 1941 [8].

Geometry of the problem. Let $\hat{z}$ be the axis of rotation of the vessel containing ^{3}He-A. Vector $\mathbf{d}$ of magnetic anisotropy is confined by the magnetic field to horizontal plane $\mathbf{d} \perp \hat{z}$. Furthermore, we can safely assume that $\mathbf{d} = \mathbf{d}_0$ is everywhere constant which reduces the order parameter to its orbital part $\Psi = \mathbf{e}_1 + i\mathbf{e}_2$, where $\mathbf{e}_1 \perp \mathbf{e}_2$, $\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1$. This simplification does not alter the topological classification because d-vector has no net winding neither inside the soliton nor inside vortices under consideration [6]. The remaining manifold of degenerate states is thus SO_3. Tiny spin-orbit interaction tends to fix the vector of orbital anisotropy $\mathbf{l} = \mathbf{e}_1 \times \mathbf{e}_2$ to $\pm\mathbf{d}_0$, restricting the degeneracy space to $S^1 \times Z_2$. Nontriviality of relative homotopy groups $\pi_2(SO_3; S^1 \times Z_2) = Z$ and $\pi_1(SO_3; S^1 \times Z_2) = Z_2$ gives rise to a possibility of non-singular 4π-vortices and solitons [4, 9]. In the geometry of Helsinki experiment magnetic field and vortices are oriented along $\hat{z}$ while the soliton plane is perpendicular to $\hat{z}$ which leads to a particular type of soliton called twist soliton [10]. The vector $\mathbf{l}$ rotates from $\mathbf{d}_0$ to $-\mathbf{d}_0$ inside the soliton:

$$\mathbf{l}_{sol} = \mathbf{U}_z \mathbf{d}_0, \tag{1}$$

where $\mathbf{U}_z$ is the matrix of rotation in xy-plane by the angle $\alpha(z)$ increasing from 0 to π. The whole sphere is swept by the vector $\mathbf{l}$ in any horizontal cross-section of a 4π-vortex, while far from the vortex $\mathbf{l}$ becomes constant. A possible ansatz is as follows:

$$\mathbf{l} = \mathbf{U}_z \left(\hat{y} \sin \eta(r) + \cos \eta(r)[\hat{x} \sin(\varphi - \varphi_0(z)) - \hat{z} \cos(\varphi - \varphi_0(z))]\right), \tag{2}$$

where r, φ are polar coordinates in xy-plane; $\eta(r)$ varies from $\eta(0) = -\pi/2$ to $\eta(\infty) = \pi/2$, $\varphi_0(z)$ is some function of z, and we have chosen $\mathbf{d}_0 = \hat{y}$.

Consider loci $C_\uparrow$, $C_\downarrow$ of points at which the vector $\mathbf{l}$ equals $\pm\hat{z}$ respectively. Depending upon the function $\varphi_0(z)$ they interlace or do not interlace (Fig.1). This notion of interlacing has a topological sense, and the distinction between these two possibilities can in principle be resolved in the experiment. Here we have implicitly supposed that the values $\mathbf{l}(\mathbf{r}) = \pm\hat{z}$ are in generic position, so that $C_\uparrow$ are $C_\downarrow$ are closed curves, possibly made of several disjoint links. The same assumption will be made below when we will also be considering loci $C_{\mathbf{l}_0}$ of points $\mathbf{r}$ at which $\mathbf{l}(\mathbf{r}) = \mathbf{l}_0$ with arbitrary $\mathbf{l}_0$.

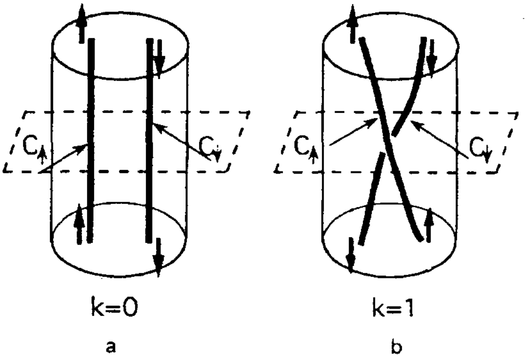

Fig.1. Topology of two different structures of a vortex crossing soliton (before duplication). *a* –Loci $C_\uparrow$ and $C_\downarrow$ at which $\mathbf{l} = \pm\hat{z}$ do not interlace; *b* – interlace

Formal description: low rotational velocities. Following [6] we duplicate the soliton, which gives us periodic boundary conditions in z-direction, and embrace the region of intersection by a large cylinder (Fig.2*a*). If the density of vortices in the vortex lattice is small, which corresponds to low rotational velocities, then each individual vortex is well-defined, permitting us to impose constant boundary conditions at the boundary ∂D^2 of any cross-section D^2 of the cylinder. With constant boundary conditions each cross-section effectively becomes a 2-sphere S^2, and the whole volume inside the cylinder should be thought of as a direct product $S^2 \times S^1$ of this sphere and $\hat{z}$-axis circumference. In addition, the vector $\mathbf{l}$ on the lateral surface of the cylinder is confined to xy-plane, as it should be for the soliton structure, restored far from the vortex. As the density of vortices increases, the vector $\mathbf{l}$ starts to deviate from xy-plane. Finally, at large densities (large rotational velocities) the intervortex distance and the size of an individual vortex become of the same order. We can no longer assume constant boundary conditions in cross-sections, but rather periodic ones. Each cross-section of the cylinder effectively becomes a torus $S^1 \times S^1$, and the whole cylinder will be $S^1 \times S^1 \times S^1$.

Note, that from three degrees of freedom of $\Psi = \mathbf{e}_1 + i\mathbf{e}_2$ we are considering only two carried by $\mathbf{l}$. We are prompt to do this, because periodic boundary conditions are imposed only on $\mathbf{l}$ [11], and as a matter of fact one can show that the third component does not alter the topological classification.

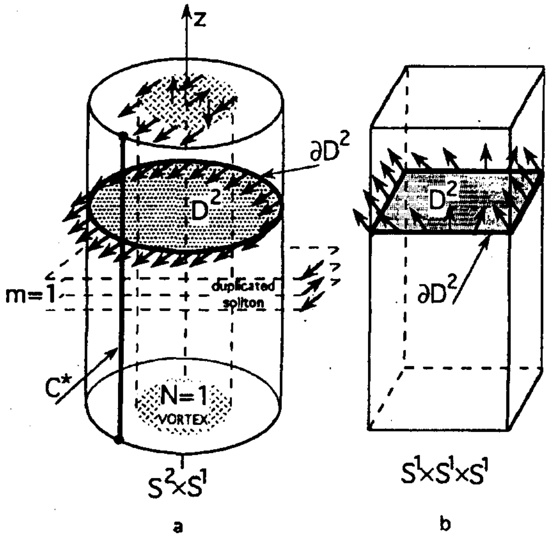

Fig.2. *a* – We embrace the region of the vortex-soliton intersection by a cylinder. Periodic boundary conditions are imposed along z. Sketched is the case with $N = 1$, $m = 1$. C^* is an arbitrary vertical generatrix of the lateral surface. For low density of vortices $\mathbf{l}$ is constant at each boundary circumference ∂D^2, therefore the cylinder effectively becomes $S^2 \times S^1$. *b* – For higher densities boundary conditions on ∂D^2 are periodic, and the cylinder becomes $S^1 \times S^1 \times S^1$. Represented is the case with zero additional invariants m_1, m_2

Now let us start the classification of possible structures. We will express invariants in terms of the superfluid velocity $\mathbf{v}_s = (\kappa/2\pi)\mathbf{e}_1\nabla\mathbf{e}_2$, related to the vector $\mathbf{l}$ by Mermin–Ho relation [12]

$$\mathrm{rot}\,\mathbf{v}_s = (\kappa/4\pi)\epsilon_{ijk} l_i \,\nabla l_j \times \nabla l_k. \tag{3}$$

Here $\kappa = \pi\hbar/m_3$ is the circulation quantum for superfluid ^{3}He. First, consider the case of $\mathbf{l}$ confined to xy-plane at the boundary of the cell. This case corresponds to the classification of all mappings of pairs $(D^2 \times S^1; \partial D^2 \times S^1) \to (S^2; S^1)$, i.e., the mappings $D^2 \times S^1 \to S^2$, $\mathbf{r} \mapsto \mathbf{l}(\mathbf{r})$ that map the boundary $\partial D^2 \times S^1$ of the cell to the equator S^1 of $\mathbf{l}$-sphere. We restrict ourselves to the case in which the boundary ∂D^2 of a cross-section of the cell is not wound over the equator by the mapping $\mathbf{l}(\mathbf{r})$ (this means that the corresponding mapping $\partial D^2 \to S^1$ is topologically trivial). This is the case for both constant (S^2) and periodic $(S^1 \times S^1)$ boundary conditions. In this case the distribution of the vector $\mathbf{l}$ in the cell is described by three integer topological invariants:

1) the number N of 4π-vortices inside the cylinder, (usually $N = 1$), given by

$$N = \frac{1}{2\kappa} \oint_{\partial D^2} \mathbf{v}_s d\mathbf{r}, \tag{4}$$

where the integral is taken along the boundary of a horizontal cross-section;

2) the algebraic number m of solitons before duplication (i.e., the number of solitons with a given direction of twist minus the number of solitons with the opposite direction of twist) that shows how many times the vector $\mathbf{l}$ goes around the equator of the sphere as $\mathbf{r}$ goes along any vertical generatrix of the lateral surface of the cylinder;

3) the linking number k of $C_\uparrow$ and $C_\downarrow$.

Numbers k and m correspond to that introduced in [6]. The linking number k is analogous and closely connected to the usual Hopf invariant of mappings $S^3 \to S^2$, given by $H = (1/4\kappa^2)\int \mathbf{v}_s \operatorname{rot} \mathbf{v}_s dV$ [13, 1]. It turns out that the expression for k in terms of the superfluid velocity suggested in [6] is valid in this case. For arbitrary N it takes the form:

$$k = \frac{1}{4\kappa^2}\int \mathbf{v}_s \operatorname{rot} \mathbf{v}_s dV - \frac{N}{2\kappa}\oint_{C^*} \mathbf{v}_s d\mathbf{r}, \tag{5}$$

and contains an additional integral over an arbitrary vertical generatrix C^* of the lateral surface (Fig.2a) of the cell.

Intermediate rotational velocities. The vector $\mathbf{l}$ is no longer necessarily horizontal nor constant at the boundary of any cross-section, the only restriction being that it is not vertical ($\mathbf{l} \neq \pm\hat{z}$). Boundary conditions become periodic with vortex lattice periods. Nevertheless, constant boundary conditions will also be considered as a subcase that might occur relevant in some other physical situation. For constant boundary conditions this case is equivalent to the previous one from the topological point of view, i.e., mappings are described by indices N, m and k. Under periodic conditions two more invariants m_1, m_2 arise that are winding numbers of the horizontal projection of the vector $\mathbf{l}$ along two non-parallel sides of the boundary ∂D^2 of a unit cell of the lattice (Fig.2b). In order to give analytical expressions for k we need the notion of the area enclosed by a contour C on the $\mathbf{l}$-sphere.

This notion is apparently ill-defined. Still one can take any unit vector $\mathbf{n}$ such that C does not contain $\mathbf{n}$ and define the area $\mathrm{Area}|_C^{\mathbf{n}}$ with respect to this vector to be $-\int_C (1 + (\mathbf{l}\mathbf{n}))d\phi/4\pi$, where ϕ is the longitude (the angle of rotations around $\mathbf{n}$). It gives the area swept by the arc of the great circle connecting $-\mathbf{n}$ and $\mathbf{l}$ that does not contain $\mathbf{n}$ (with the orientation taken into account). One can show that this quantity does not depend on $\mathbf{n}$ as it moves over the sphere unless it crosses C. At this moment the integral changes by an integer. It follows that only the fractional part of the area enclosed by C (in the units of the whole area of the sphere) has an invariant meaning (we designate this fractional part $\mathrm{Area}|_C$). Let C^* be a contour in the real space, and $\mathbf{l}(C^*)$ its image on S^2. We will denote $\mathrm{Area}|_{\mathbf{l}(C^*)}$ by $\mathrm{Area}|_{C^*}$ as well.

After these preliminaries we can write down a more general formula for k:

$$k = \frac{1}{2\kappa}\oint_{C_\uparrow} \mathbf{v}_s d\mathbf{r} - \frac{N}{2\kappa}\oint_{C^*} \mathbf{v}_s d\mathbf{r} + N\,\mathrm{Area}|_{C^*}^{\downarrow}, \tag{6}$$

where $\downarrow$ denotes the south pole. An analogous formula with $\uparrow$ and $\downarrow$ interchanged is also valid. For $\mathbf{l}$ lying in the xy-plane at the boundary $\mathrm{Area}|_{C^*}^{\downarrow} = -\mathrm{Area}|_{C^*}^{\uparrow} = m/2$.

Let us also mention the following relation for the circulation of the superfluid velocity along $C_\mathbf{l}$ averaged over the whole sphere of vector $\mathbf{l}$:

$$\left\langle \oint_{C_\mathbf{l}} \mathbf{v}_s d\mathbf{r} \right\rangle = \frac{1}{2\kappa}\int \mathbf{v}_s \operatorname{rot} \mathbf{v}_s dV. \tag{7}$$

For the case of the vector $\mathbf{l}$ horizontal at the boundary the integral over $C_\mathbf{l}$ in the left-hand side is constant on the northern and south hemispheres, which permits to link (6) and (5) with the aid of (7).

Here is another equivalent formula for k, through $\int \mathbf{v}_s \,\mathrm{rot}\, \mathbf{v}_s dV$ (under the assumption $\mathbf{l} \neq \pm\hat{z}$ at the boundary):

$$k = \frac{1}{4\kappa^2} \int \mathbf{v}_s \,\mathrm{rot}\, \mathbf{v}_s dV - \frac{N}{\kappa} \oint\limits_{C^*} \mathbf{v}_s d\mathbf{r} + \frac{N}{2\kappa} \left\langle \oint\limits_{C^{**}} \mathbf{v}_s d\mathbf{r} \right\rangle + N \left(\mathrm{Area}|^{\downarrow}_{C^*} + \mathrm{Area}|^{\uparrow}_{C^*} \right). \quad (8)$$

Here C^* is a fixed vertical generatrix of the lateral boundary, C^{**} is an arbitrary vertical generatrix of the boundary and the angle brackets denote the averaging over all such generatrices. To derive (8) the use was made of the periodicity of $\mathbf{v}_s - \mathbf{\Omega} \times \mathbf{r}$ [11].

Large rotational velocities. Suppose now that the angular velocity is considerably large and $\mathbf{l}$ may take arbitrary values at the boundary. The previous classification fails as winding numbers m, m_1, m_2 of the vector $\mathbf{l}$ cannot be defined. Depending on whether we assume constant or periodic boundary conditions we should now classify the mappings $S^2 \times S^1 \to S^2$ or $S^1 \times S^1 \times S^1 \to S^2$ (Fig.2). It can be shown (first done in [8]) that a mapping $S^2 \times S^1 \to S^2$ is characterized by a number N (the number of quanta or the degree of the mapping of horizontal cross-section to l-sphere), and, for given N, by an element of Z_{2N} (i.e., integer modulo $2N$). For the mappings in the classes discussed before this element is given by $I = k + mN \pmod{2N}$. This means that configurations with the same N and $k + mN \pmod{2N}$ are topologically equivalent if we allow $\mathbf{l}$ to take arbitrary values at the boundary. In particular, for $N = 1$ the invariant is $k + m \pmod 2$, i.e., there are two types of configurations of the intersection of 4π-vortex with a soliton. For example, the configurations $m = k = 1$ and $m = 1$, $k = 0$ mentioned in [6] are topologically different even in the limit of large angular velocities (this is true for periodic boundary conditions as well, see below). For $N = 0$ the invariant is an integer k: a localized defect in a uniform field of vector $\mathbf{l}$ ($m = 0$) or inside a twisted soliton ($m = 1$) is characterized by the linking number of two loops $C_{\mathbf{l}_1}$ and $C_{\mathbf{l}_2}$.

Fix any $\mathbf{l}_0$ such that the field of the vector $\mathbf{l}$ does not take the value $\mathbf{l}_0$ at C^* (i.e., at the boundary). Then the invariant I is given by

$$I = \frac{1}{2\kappa} \oint\limits_{C_{\mathbf{l}_0}} \mathbf{v}_s d\mathbf{r} - \frac{N}{2\kappa} \oint\limits_{C^*} \mathbf{v}_s d\mathbf{r} + N \mathrm{Area}\Big|^{\mathbf{l}_0}_{C^*} \pmod{2N}, \quad (9)$$

or, in terms of $\int \mathbf{v}_s \,\mathrm{rot}\, \mathbf{v}_s dV$:

$$I = \frac{1}{4\kappa^2} \int \mathbf{v}_s \,\mathrm{rot}\, \mathbf{v}_s dV - \frac{N}{2\kappa} \oint\limits_{C^*} \mathbf{v}_s d\mathbf{r} + 2N \mathrm{Area}|_{C^*} \pmod{2N}. \quad (10)$$

Mappings $S^1 \times S^1 \times S^1 \to S^2$ are described in general by three integers N_1, N_2, N_3 which show how many times three different faces of the cell cover the l-sphere, and by an element of $Z_{2\cdot \mathrm{GCD}(N_1, N_2, N_3)}$ where GCD denotes the greatest common divisor of N's. It follows, however, from periodicity of $\mathbf{v}_s - \mathbf{\Omega} \times \mathbf{r}$ [11] that for the configuration in question only one of three indices N_i is nonzero, corresponding to horizontal face. We denote it by $N = \mathrm{GCD}(N, 0, 0)$. The formula for the invariant is as follows (cf. (8)):

$$I = \frac{1}{4\kappa^2} \int \mathbf{v}_s \,\mathrm{rot}\, \mathbf{v}_s dV - \frac{N}{\kappa} \oint\limits_{C^*} \mathbf{v}_s d\mathbf{r} + \frac{N}{2\kappa} \left\langle \oint\limits_{C^{**}} \mathbf{v}_s d\mathbf{r} \right\rangle + 2N \mathrm{Area}|_{C^*} \pmod{2N}. \quad (11)$$

To conclude, we have discussed the topology of the intersection of a lattice of 4π-vortices and a soliton in ^{3}He-A and derived the analytical expressions for the proper invariants in terms of the distributions of the vector $\mathbf{l}$ and the superfluid velocity. We have proved using these invariants that the two competing configurations considered in [6] remain topologically different under any rotational velocity. The results also apply to other media with broken symmetry described by a 3D vector, like Heisenberg ferromagnets, subjected to periodic or mixed periodic-constant boundary conditions.

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