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DYNAMICAL FLUCTUATIONS OF VESICLES WITH NONTRIVIAL TOPOLOGY

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The role of conformal degeneracy of the Helfrich energy for shape fluctuations of vesicles is discussed. For a vesicle with a nontrivial topology (genus $g > 1$) there exists a two-parametric set of transformations of the shape which does not influence the Helfrich energy, surface and volume of the vesicle [4]. We have shown that the region $\mathcal{L}$ of the phase space corresponding to the transformations is finite. Higher order corrections to the Helfrich energy which breaks the degeneracy are small in comparison with the temperature T. It leads to the equipartition over the region $\mathcal{L}$. The characteristic time of the shape fluctuations of a vesicle is of the order of $\eta R^3/T$, where T is the temperature, η is the viscosity of the liquid surrounding the vesicle and R is its size.

Recently vesicles (closed structures constructed from membranes that is bilayer films) with non-trivial topology (genus $g > 1$) have been observed experimentally [1]. The characteristic feature of such vesicles is the existence of strong shape fluctuations, much larger than conventional bending fluctuations. To explain this behavior the authors of the paper [2] have analyzed consequences of the conformal degeneracy of the bending energy. They have used the numerical simulation for a discretized version of the bending energy for so-called Lawson's surfaces with $g = 2$. Their numerical results show the existence of a certain region in the phase space where the volume and the area of vesicles are kept constant. Just this property leads to strong shape fluctuations for Lawson's vesicles. Here we are going to investigate general physical consequences of this conformal degeneracy for vesicles with $g \geq 2$.

The incompressibility of the bilayer and of the liquid contained in a vesicle imposes two constraints on the fluctuations of the vesicle shape which are the area S and the volume V conservation laws. The fluctuations within the constraints are governed by the Helfrich (bending) energy of a bilayer [3]

$$F = \frac{\kappa}{2}\int dS\left(\frac{1}{R_1}+\frac{1}{R_2}\right)^2 , \tag{1}$$

where R_1, R_2 are the local curvature radii of the film. The combination in the brackets is the double mean curvature. Generally speaking the term with the Gaussian curvature $(R_1R_2)^{-1}$ should be added to (1), but the integral of the Gaussian curvature over the surface is proportional to its genus. Thus the term is not influenced by fluctuations which do not change the topology of the vesicle. We will consider only such fluctuations and therefore will not take the term with the Gaussian curvature into account.

The Helfrich energy (1) is conformally invariant [4]. It means that any element of the conformal group transforms a vesicle shape into a new one with the same energy (1). The total symmetry group of the Helfrich energy (1) contains also translations, rotations, reflections and dilations. It is clear that a deviation of a vesicle shape from the ground state relating to a transformation of the group and consistent with both constraints $V = \mathrm{const}$ and $S = \mathrm{const}$ would be very soft one because in the main approximation it does not change the thermodynamical potential of a vesicle. In the following we will call such fluctuations conformal ones.

Any transformation $\mathbf{r} \rightarrow \mathbf{r}'$ from the group could be represented as a combination of a conformal transformation with the center at the origin

$$\frac{\mathbf{r}'}{r'^2} = \frac{\mathbf{r}}{r^2} + \mathbf{a} \quad , \tag{2}$$

of a dilation $\mathbf{r}' = \lambda \mathbf{r}$ and of a translation, a reflection and a rotation. Certainly translations, rotations and reflections do not change the shape of the vesicle. Thus in the general case we have the four-parametric set of nontrivial transformations of the vesicle shape which do not influence the energy (1), these parameters could be thought as three components of a vector a determined by (2) and one scaling parameter λ describing a dilation. From the first sight it seems that we might always satisfy two constraints $S = \mathrm{const}$ and $V = \mathrm{const}$ and retain two free parameters for conformal fluctuations. However this statement is not true for spheres and Clifford torii (torii with the ratio of the principal radii $1/\sqrt{2}$) providing a minimum of (1) with $g = 0$ and $g = 1$ correspondingly.

Indeed, a dilation or a conformal transformation of a sphere produces a sphere again. Therefore the four-parametric set of transformations in this case is reduced to the one-parametric set described by the radius of the sphere. The constraint V =const determines the radius of this sphere unambiguously. For a Clifford torus the dilation gives a Clifford torus again. Any conformal transformation (2) could be represented as a result of two transformations with the vectors $\mathbf{a}_{\parallel}$ parallel to the axis of the torus and $\mathbf{a}_{\perp}$ perpendicular to it since the conformal group is an Abelian one. The conformal transformation of a Clifford torus with $\mathbf{a}_{\parallel}$ produces a Clifford torus again. Thus the four-parametric set of transformations in this case is reduced to the two-parametric one describing by the size of the Clifford torus and by the absolute value of $\mathbf{a}_{\perp}$. Both the parameters are fixed by the constraints $V = \mathrm{const}$ and $S = \mathrm{const}$. Thus in the cases $g = 0$ and $g = 1$ the minimum of (1) with two constraints $S = \mathrm{const}$ and $V = \mathrm{const}$ fixes the shape of the vesicle unambiguously, it being the consequence of the high symmetry of spheres and Clifford torii. For vesicles with $g \geq 2$ the two-parametric set of conformal transformations exists. It describes introduced above conformal fluctuations. For Lawson's surfaces with $g = 2$ it has been shown numerically in [2].

One can expect completely different consequences whether the region $\mathcal{L}$ of conformal deformations in the two-dimensional parameter space is finite or not. Let us suppose for a moment that this region $\mathcal{L}$ is infinite. It means e. g. that there exists a combination of a dilation and of a transformation (2) conserving S and V with unrestricted value of a. One can represent this combination as

$$\frac{\mathbf{r}'}{r'^2} = \frac{1}{\lambda}\left(\frac{\mathbf{r}}{r^2} + \mathbf{a}\right) \quad . \tag{3}$$

For large $|\mathbf{a}|$ (3) is reduced to

$$r_i' - \frac{a_i}{a^2} = \frac{\lambda}{a^2}\left(\delta_{ik} - 2\frac{a_i a_k}{a^2}\right)\frac{r_k}{r^2} \quad . \tag{4}$$

This expression is the principal term of the expansion of (3) in the limit $a \gg r^{-1}$. The transformation (4) is the product of the translation (the second term in the l.h.s. of (4)), of the dilation (the first factor in the r.h.s.), of the reflection of the coordinate along the vector a (the combination in the brackets) and of the inversion with the center at the origin (the third factor in the r.h.s.). Since the reflection keeps V and S we have only one free parameter in (4), namely, λ/a^2, and it is impossible to satisfy both constraints $V = \text{const}$ and $S = \text{const}$. Therefore our assumption leads to the contradiction what proves that the region $\mathcal{L}$ should be finite. It is reasonable from the physical point of view since it is hard to imagine a strong deformation of the vesicle shape conserving V, S and the energy (1).

The conformal degeneracy of the bending energy is broken by higher order terms:

$$F_a = \gamma \int dS (\nabla_\perp (R_1^{-1} + R_2^{-1}))^2 + \dots \quad , \tag{5}$$

where $\nabla_\perp$ is the gradient along the film. We have written in (5) explicitly only one term, other contributions denoted in (5) by dots have structures similar to one of the term and can be evaluated identically. The natural estimation for the coefficient γ is

$$\gamma \sim \kappa a_m^2 \quad ,$$

where a_m is the typical molecular scale. The characteristic scale of the conformal fluctuations is of the order of the size R of the vesicle and therefore for conformal fluctuations the energy (5) is

$$F_a \sim \kappa \left(\frac{a_m}{R}\right)^2 \quad .$$

Usually the modulus κ is larger than the temperature T but does not much exceed it (as a rule it is of the order of $T \simeq \kappa/10$) and $a_m \ll R$. Thus the energy F_a is much smaller than the temperature. It means that the character of conformal fluctuations is not sensitive to F_a.

Thus the conformal fluctuations are governed only by thermal noise what leads to the equipartition over the allowed region $\mathcal{L}$ in the phase space. Using the equipartition we can estimate the mean-square amplitude of the shape fluctuations related to the conformal mode

$$\langle (\Delta R)^2 \rangle \sim R^2 \quad . \tag{6}$$

Let us emphasize that the crucial fact for this conclusion is the existence just a finite region $\mathcal{L}$ in the phase space where conformal transformations take place. Expression (6) shows that the amplitude of conformal fluctuations is much larger than one of conventional bending fluctuations which are weak due to the smallness of T/κ.

To investigate the dynamical behavior of conformal fluctuations one should analyze the hydrodynamic motion of the liquid near the vesicle. There exist specific degrees of freedom associated with variations of the shape of the vesicle including ones associated with conformal fluctuations. The latters are the most soft ones and to describe their dynamical behavior one can use some kind of adiabatic approximation. The rigorous procedure to investigate dynamical fluctuations is described in our monograph [5] (see also the paper [6] especially devoted to bending fluctuations of membranes). However qualitative results concerning conformal fluctuations can be deduced without using this rather cumbersome technique. Since the bending energy in the region $\mathcal{L}$ is negligibly small we deal with the passive motion or with the advection of the vesicle shape over the region $\mathcal{L}$. Properties of the motion are determined by the competition of the thermal noise and the viscous forces what means that the characteristic time t of the conformal fluctuations is

$$t \sim \eta R^3/T \quad . \tag{7}$$

To derive the estimation one should remind that according to (6) the characteristic amplitude of the conformal fluctuations is just of the order of R. One can call dynamics of the fluctuations the conformal diffusion since the estimation (7) for t is no other than the conventional Einstein diffusion time (see e. g. [7]).

In summary we have shown that fluctuations of the shape of a vesicle with $g \geq 2$ are associated with the conformal degeneracy of the Helfrich energy. They can be thought as the diffusion over the finite region $\mathcal{L}$ in the phase space, this diffusion is characterized by (6, 7).

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