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ON DEFORMATIONS OF SOME INTEGRABLE EQUATIONS

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We show that deformations of integrable equations in $1 \leq d \leq 3$ or, in other words, nonautonomous versions of well-known integrable equations can be obtained by reduction of the self-duality equations of the Yang-Mills model in $d = 4$ under the action of symmetry groups. We describe new nonautonomous integrable equations (and their linear systems), which are the deformations of the equation of the principal chiral model in $d = 3$, the Korteweg-de Vries equation and the equations of the Hamiltonian systems with quartic potentials.

1. Most of the integrable equations in $(1+1)$ dimensions arise as compatibility conditions of the overdetermined linear system of equations [1, 2]

$$(\partial_t + V)\Psi = 0, \quad (\partial_x + U)\Psi = 0. \tag{1}$$

Here V and U are matrices from the algebra $gl(n, C)$ which depend on the coordinates t, x and on a constant complex spectral parameter λ, and $\Psi \in C^n$ is a vector-function depending on t, x and λ. However, it has been shown [3, 4] that the Ernst equation arises as a compatibility condition of a linear system, more general than (1). Belinsky and Zakharov have generalized (1) by adding terms with a derivative with respect to the spectral parameter λ [3]. They have also generalized the inverse scattering transform method for solving that type of systems. At the same time, Maison has written out the linear system for the Ernst equation in the form of (1) but with the spectral parameter λ which is the function of t and x and of a hidden spectral parameter [4]. The dressing method for this linear system has been developed by Mikhailov and Yaremchuk [5]. They also constructed the explicit solutions and investigated the conservation laws. Later on, Burtsev, Zakharov and Mikhailov have shown [6] that both generalizations of the linear system (1) are equivalent and developed the general method of integration of nonlinear equations arising as compatibility conditions of such generalized linear systems. Moreover, they have shown that with the help of the above-described generalization of the linear system (1) to each equation in $(1+1)$ dimensions, integrable by the inverse scattering transform method, one can associate some integrable equations with the coefficients depending on t and x. They called these equations the deformations of initial equations.

On the other hand, L.Witten has shown [7] (see also [8]) that the Ernst equation can be obtained by reducing the Self-Dual Yang-Mills (SDYM) equations that have been introduced in the famous paper of Belavin, Polyakov, Schwarz and Tyupkin [9]. The integrability of the SDYM equations has been proved by Belavin and Zakharov [10] and Ward [11]. Now it is known [12, 8] that the reduction of the SDYM equations under two translations yields an equation of the principal

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chiral model in $d=2$, and the reduction of the SDYM equations under translation and rotation yields a deformed equation of the principal chiral model which, in a particular case of the gauge group $GL(2,R)$, is equivalent to the Ernst equation.

Later, it has been shown that not only the equation of the principal chiral model in $d=2$, but also many other integrable equations in $1 \leq d \leq 3$ dimensions may be obtained by reductions of the SDYM equations with respect to the action of subgroups of the group of translations in R^4 (see, e.g., [12–14] and references therein). It is natural to suppose that the replacement of some generators of the translation group by generators of the rotation group will permit us to obtain, by symmetry reduction of the SDYM equations, the deformation not only of the equation of the principal chiral model in $d=2$ but also of some other integrable equations. The purpose of this Letter is to prove this supposition.

We show that the deformed nonlinear Schrödinger equation (NLS), considered in [6, 15], can be deduced by reducing the SDYM equations. We also describe three new examples of nontrivial deformations of the well-known integrable equations (and their linear systems), which can be obtained by reductions of the SDYM equations under different symmetry groups. Among them there are deformations of the equation of the principal chiral model in $R^{2,1}$, considered by Manakov and Zakharov [16], the Korteweg–de Vries (KdV) equation and the equations of the Hamiltonian systems with quartic potentials.

2. We consider the space $R^{2,2}$ with the metric $(g_{\mu\nu}) = diag(+1,+1,-1,-1)$ and the potentials A_μ of the Yang–Mills (YM) fields $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, where $\mu, \nu, ... = 1, ..., 4$, $\partial_\mu = \frac{\partial}{\partial x^\mu}$. The fields A_μ and $F_{\mu\nu}$ take values in the Lie algebra $gl(n,C)$.

In $R^{2,2}$, we introduce the null coordinates

$$t = \frac{1}{2}(x^2 - x^4), \quad u = \frac{1}{2}(x^2 + x^4), \quad y = \frac{1}{2}(x^1 - x^3), \quad z = \frac{1}{2}(x^1 + x^3)$$

and set $A_t = A_2 - A_4$, $A_u = A_2 + A_4$, $A_y = A_1 - A_3$, $A_z = A_1 + A_3$. The SDYM equations in the null coordinates have the following form:

$$F_{tz} = 0, \quad F_{uy} = 0, \quad F_{tu} + F_{zy} = 0. \tag{2}$$

Equations (2) can be obtained as compatibility conditions of the following linear system of equations (cf. ref. [10, 11]):

$$(\partial_t - \lambda\partial_y + A_t - \lambda A_y)\Psi = 0, \tag{3a}$$

$$(\partial_z + \lambda\partial_u + A_z + \lambda A_u)\Psi = 0, \tag{3b}$$

$$\partial_{\bar{\lambda}}\Psi = 0, \tag{3c}$$

where $\bar{\lambda}$ is the complex conjugate to λ. Here Ψ is a column vector depending on the coordinates of $R^{2,2}$ and on the "coordinates" λ and $\bar{\lambda}$, parametrizing the upper sheet of the hyperboloid $H^2 = SO(2,1)/SO(2)$. Notice that Ψ is defined on the twistor space $\mathcal{Z} = R^{2,2} \times H^2$ for the space $R^{2,2}$ and eqs.(3) mean the holomorphicity of the vector-function Ψ (Ward theorem [11]).

3. We consider the inhomogeneous group of rotations $ISO(2,2)$ (rotations and translations) and an arbitrary subgroup G of the group $ISO(2,2)$. We would like to impose the conditions of G-invariance on the potentials A_μ and on the vector-function Ψ. For that, we have to define the generators of the group $ISO(2,2)$

as vector fields on $R^{2,2}$, when considering the action of G on A_μ, and as vector fields on the twistor space $R^{2,2}\times H^2$, when considering the action of G on Ψ [17].

Let us introduce the following constant tensors:

$$f^a_{\mu\nu}=\{f^a_{bc},\ \mu=a,\nu=b;\ \delta^a_\mu,\nu=4;\ -\delta^a_\nu,\mu=4\},\quad I_a{}^\mu_\nu=-\frac{1}{2}g_{ab}g^{\mu\lambda}f^b_{\lambda\nu}, \tag{4a}$$

$$\bar f^a_{\mu\nu}=\{f^a_{bc},\ \mu=a,\nu=b;\ -\delta^a_\mu,\nu=4;\ \delta^a_\nu,\mu=4\},\quad J_a{}^\mu_\nu=-\frac{1}{2}g_{ab}g^{\mu\lambda}\bar f^b_{\lambda\nu}, \tag{4b}$$

where $a,b,...=1,2,3$, $g_{11}=g_{22}=-g_{33}=1$ and $f^1_{23}=f^2_{31}=-f^3_{12}=1$ are the structure constants of the group $SO(2,1)$. Then, the generators of the group $ISO(2,2)$ can be realized in terms of the following vector fields on $R^{2,2}$:

$$X_a=I_a{}^\mu_\nu x^\nu\partial_\mu,\quad Y_a=J_a{}^\mu_\nu x^\nu\partial_\mu,\quad P_\mu=\partial_\mu. \tag{5}$$

The vector fields on $Z=R^{2,2}\times H^2$, which also form the generators of $ISO(2,2)$, are given by

$$\tilde X_a=X_a,\quad \tilde Y_a=Y_a+Z_a,\quad \tilde P_\mu=P_\mu, \tag{6a}$$

with the following expression of the generators Z_a of the $SO(2,1)$-rotations on H^2

$$Z_1=\frac{1}{2}[(1-\lambda^2)\partial_\lambda+(1-\bar\lambda^2)\partial_{\bar\lambda}],\ Z_2=-[\lambda\partial_\lambda+\bar\lambda\partial_{\bar\lambda}],\ Z_3=-\frac{1}{2}[(1+\lambda^2)\partial_\lambda+(1+\bar\lambda^2)\partial_{\bar\lambda}]. \tag{6b}$$

It can be easily shown that $[X_a,X_b]=f^c_{ab}X_c$, $[Z_a,Z_b]=f^c_{ab}Z_c$ and so on.

To reduce the SDYM equations (2) and the linear system (3) under a subgroup G of the group $ISO(2,2)$, it is necessary to impose the following conditions of G-invariance on the potentials A_μ and on the vector-function Ψ [18]:

$$W_\xi A_\mu+A_\sigma W^\sigma_{\xi,\mu}=0,\quad \forall\xi\in\mathcal{G}, \tag{7a}$$

$$\tilde W_\xi\Psi=0,\quad \forall\xi\in\mathcal{G}, \tag{7b}$$

where $\mathcal{G}$ is the Lie algebra of the group G, $W_\xi=W^\sigma_\xi\partial_\sigma$ are the vector fields on $R^{2,2}$ and $\tilde W_\xi$ are the vector fields on $R^{2,2}\times H^2$. Both W_ξ and $\tilde W_\xi$ form a realization of the Lie algebra $\mathcal{G}$.

In accordance with the general method of symmetry reduction (see [19] and references therein), as new coordinates on $R^{2,2}$, one should choose the coordinates θ_ξ on the orbits of the group G, and the invariant coordinates θ_A ($A=1,...,4-dimG$) and ζ which parametrize the space of orbits and satisfy:

$$\tilde W_\xi\theta_A=0,\quad \tilde W_\xi\zeta=0,\quad \partial_{\bar\lambda}\zeta=0,\quad \forall\xi\in\mathcal{G}. \tag{8}$$

Here, the invariant complex coordinate ζ represents the new "spectral parameter". Then, substituting solutions of eqs.(7) and (8) into eqs.(2), (3), we obtain the reduced SDYM equations and their linear system in terms of functions of the invariant coordinates [17, 19].

4. *Deformed chiral model equation in* $R^{2,1}$. We consider the one-dimensional group of rotations generated by the vector field X_2+Y_2. From (5) and (6) we obtain $\tilde X_2+\tilde Y_2=X_2+Y_2+Z_2=z\partial_z-y\partial_y-\lambda\partial_\lambda-\bar\lambda\partial_{\bar\lambda}$. On $R^{2,2}\times H^2$ let us introduce the coordinates ρ,θ,η,ξ by formulae $y=\frac{1}{2}\rho e^{-\theta}$, $z=\frac{1}{2}\rho e^{\theta}$, $\lambda=\eta e^{i\xi}$,

then $X_2 + Y_2 = \partial_\theta$ and $\tilde{X}_2 + \tilde{Y}_2 = \partial_\theta - \eta\partial_\eta$. Therefore, $\varphi = \frac{1}{2}(\theta - \ln\eta)$ will be the coordinate on the orbit and $t, u, \rho, \zeta = \lambda e^\theta$ will be the invariant coordinates.

The invariant YM potentials A_μ, satisfying eqs.(7a), have the form

$$A_t = T_t(t,u,\rho), \quad A_u = T_u(t,u,\rho), \quad A_y = T_y(t,u,\rho)e^\theta, \quad A_z = T_z(t,u,\rho)e^{-\theta}. \tag{9a}$$

The vector-function

$$\Psi = \psi(t,u,\rho,\zeta) \tag{9b}$$

is the solution of equations (7b) and (3c).

Substituting (9) into (3), we obtain the following reduced linear system:

$$\nabla_{V_1}\psi \equiv [\partial_t - \zeta\partial_\rho + \frac{1}{\rho}\zeta^2\partial_\zeta + T_t - \zeta T_y]\psi = 0, \quad \nabla_{V_2}\psi \equiv [\partial_\rho + \zeta\partial_u + \frac{1}{\rho}\zeta\partial_\zeta + T_z + \zeta T_u]\psi = 0, \tag{10}$$

where $V_1 = \partial_t - \zeta\partial_\rho + \frac{1}{\rho}\zeta^2\partial_\zeta$, $V_2 = \partial_\rho + \zeta\partial_u + \frac{1}{\rho}\zeta\partial_\zeta$. Remind that in the general case $[V_1, V_2] \neq 0$ and then for the linear systems like (10) the compatibility condition is

$$[\nabla_{V_1}, \nabla_{V_2}] - \nabla_{[V_1,V_2]} = 0. \tag{11}$$

Correspondingly, the SDYM equations (2) are reduced to

$$\partial_t T_z - \partial_\rho T_t + [T_t, T_z] = 0, \quad \partial_\rho T_u - \partial_u T_y + [T_y, T_u] = 0, \tag{12a}$$

$$\partial_\rho(T_y - T_z) + \frac{1}{\rho}(T_y - T_z) + \partial_t T_u - \partial_u T_t + [T_t, T_u] + [T_z, T_y] = 0 \tag{12b}$$

which agree with the compatibility condition (11) of the linear system (10).

Choosing the algebraic constraints $T_z = T_t = 0$, from eqs.(12a) we obtain $T_u = g^{-1}\partial_u g, T_y = g^{-1}\partial_\rho g$. Then, eq.(12b) is reduced to the deformed chiral model equation in $R^{2,1}$

$$\partial_\rho(g^{-1}\partial_\rho g) + \frac{1}{\rho}g^{-1}\partial_\rho g + \partial_t(g^{-1}\partial_u g) = 0 \iff \frac{1}{\rho}\partial_\rho(\rho g^{-1}\partial_\rho g) + \partial_t(g^{-1}\partial_u g) = 0. \tag{13}$$

The linear system for this equation has the form (10) with $T_z = T_t = 0$.

Remark. Notice that if one uses an additional condition of invariance under $P_t + P_u : (\partial_t + \partial_u)\psi = (\partial_t + \partial_u)g = 0$ in the linear system (10) and in eq.(13), then one obtains the deformed equation of the principal chiral model in $R^{1,1}$ [6] which, in a particular case of the gauge group $GL(2,R)$, is equivalent to the Ernst equation [8].

5. Here, we consider examples of reductions of the SDYM equations to the integrable equations in $R^{1,1}$.

Deformed NLS equation. Let us consider the two-dimensional Abelian group with the generators $\{X_2 + Y_2, P_u\}$. Then, the invariant A_μ and Ψ are given by formulae (9) where T_μ and ψ do not depend on u. The reduced linear system and reduced SDYM equations have the form

$$[\partial_t - \zeta\partial_\rho + \frac{1}{\rho}\zeta^2\partial_\zeta + T_t - \zeta T_y]\psi = 0, \quad [\partial_\rho + \frac{1}{\rho}\zeta\partial_\zeta + T_z + \zeta T_u]\psi = 0, \tag{14}$$

$$\partial_t T_z - \partial_\rho T_t + [T_t, T_z] = 0, \quad \partial_\rho T_u + [T_y, T_u] = 0, \tag{15a}$$

$$\partial_\rho(T_y - T_z) + \frac{1}{\rho}(T_y - T_z) + \partial_t T_u + [T_t, T_u] + [T_z, T_y] = 0. \tag{15b}$$

For matrices from (14) and (15) we choose the ansatz

$$T_t = \begin{pmatrix} a & \bar{b} \\ b & -a \end{pmatrix}, \quad T_u = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_y = 0, \quad T_z = \sqrt{\kappa}\begin{pmatrix} 0 & \bar{\phi} \\ \phi & 0 \end{pmatrix}, \tag{16}$$

where a, b and ϕ are arbitrary complex-valued functions of t and ρ, κ is an arbitrary real constant parameter and the bar over the letter means complex conjugation. Substituting (16) into (15), we obtain that

$$a = -i\kappa\bar{\phi}\phi - 2i\kappa \int \frac{d\rho}{\rho}\bar{\phi}\phi, \qquad b = i\sqrt{\kappa}\,(\partial_\rho\phi + \frac{1}{\rho}\phi), \tag{17}$$

and that the function ϕ has to satisfy the equation

$$i\partial_t\phi + \partial_\rho^2\phi - 2\kappa(\bar{\phi}\phi)\phi = -\frac{1}{\rho}\partial_\rho\phi + \frac{1}{\rho^2}\phi + 4\kappa\phi\left(\int \frac{d\rho}{\rho}\bar{\phi}\phi\right). \tag{18}$$

The linear system for eq.(18) can be derived via the substitution of (16) and (17) into (14).

Remark. The deformed NLS equation (18) has been considered in the paper [6]. When $\kappa = -1$, this equation is gauge equivalent to the equation of the Heisenberg ferromagnet in axial geometry. By changing the variables t, ρ and ϕ, eq.(18) can be transformed to the equation that has been introduced and integrated in [15]. Thus, the deformed NLS equation is shown to be the reduction of the SDYM equations.

Deformed KdV equation. Now we consider the generators $\{X_2 + Y_2, P_u\}$, linear system (14) and its compatibility conditions (15). For the matrices from (15) let us choose the ansatz

$$T_t = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad T_u = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad T_y = \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix}, \quad T_z = \begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix}, \tag{19}$$

where a, b, c, f, g and h are arbitrary real-valued functions of t and ρ. Substituting (19) into (15), we obtain that

$$a = \frac{1}{4}\partial_\rho f - \frac{1}{4\rho}\int \frac{d\rho}{\rho}f, \quad b = -\frac{1}{2\rho}f - \frac{1}{2\rho}\int \frac{d\rho}{\rho}f, \quad h = \frac{1}{2}f + \frac{1}{2}\int \frac{d\rho}{\rho}f,$$

$$c = \frac{1}{4}\rho\partial_\rho^2 f - \frac{1}{4\rho}f + \frac{1}{2}f^2 + (\frac{1}{4\rho} + \frac{f}{2})\int \frac{d\rho}{\rho}f, \quad g = -\frac{1}{\rho}, \tag{20}$$

and that the function f satisfies the equation

$$\partial_t f - \frac{3}{2}f\partial_\rho f - \frac{1}{4}\partial_\rho^3(\rho f) =$$

$$= \frac{1}{2\rho^2}f + \frac{1}{2\rho}f^2 - \frac{1}{4\rho}\partial_\rho f - \frac{1}{2}\partial_\rho^2 f + (\frac{1}{2}\partial_\rho f - \frac{f}{2\rho} - \frac{1}{4\rho^2})\int \frac{d\rho}{\rho}f. \tag{21}$$

The linear system for eq.(21) is deduced by inserting (19) and (20) into the linear system (14).

Remark. The deformed KdV equations have been considered in the papers [6, 15]. Equation (21) differs from ones considered in [6, 15], and it is a new deformation of the KdV equation.

6. Finally, we consider the reduction of the SDYM equations to those of the Hamiltonian systems with quartic potentials.

Deformed equations of the Hamiltonian systems with quartic potentials. Let us consider the three-dimensional Abelian subgroup of $ISO(2,2)$ generated by the vector fields $X_2 + Y_2, P_u, P_t$. It is easy to show that the invariant YM potentials and Ψ satisfying eqs.(3c) and (7b) are given by formulae (9), where T_μ and ψ depend only on ρ and do not depend on t and u. Substituting the invariant A_μ and ψ into the linear system (3) and using (7), we obtain the following reduced linear system:

$$[\zeta\partial_\rho - \frac{1}{\rho}\zeta^2\partial_\zeta - T_t + \zeta T_y]\psi = 0, \quad [\partial_\rho + \frac{1}{\rho}\zeta\partial_\zeta + T_z + \zeta T_u]\psi = 0, \tag{22}$$

Using the compatibility condition (11) for the linear system (22), we obtain the following reduced SDYM equations:

$$\frac{d}{d\rho}T_t + [T_z, T_t] = 0, \quad \frac{d}{d\rho}T_u + [T_y, T_u] = 0, \tag{23a}$$

$$\frac{d}{d\rho}(T_y - T_z) + \frac{1}{\rho}(T_y - T_z) + [T_t, T_u] + [T_z, T_y] = 0. \tag{23b}$$

Let us choose in $GL(n,C)$ the subgroups N and H so that N/H be a compact Hermitian symmetric space (for example, $N = SU(n), H = SU(n-m)\times SU(m)\times U(1)$) [20]. Let $\mathcal{N}$ and $\mathcal{H}$ be the Lie algebras of the Lie groups N and H. Then $\mathcal{N} = \mathcal{H}\oplus\mathcal{P}$ and $[\mathcal{H},\mathcal{H}]\subset\mathcal{H}, [\mathcal{H},\mathcal{P}]\subset\mathcal{P}, [\mathcal{P},\mathcal{P}]\subset\mathcal{H}$. A special feature of Hermitian symmetric spaces is the existence of an element $A\in\mathcal{H}$ such that $\mathcal{H} = \{B\in\mathcal{N} : [A,B] = 0\}$. The matrix ad_A has only three distinct eigenvalues $0, \pm i$ and $[A,\mathcal{H}] = 0, [A, X^\pm] = \pm iX^\pm$ for all $X^\pm\in\mathcal{P}^\pm, \mathcal{P} = \mathcal{P}^+\oplus\mathcal{P}^-$. Let $e_{\pm\alpha}$ be a basis of the space $\mathcal{P}^\pm$. Then $[A, e_{\pm\alpha}] = \pm ie_{\pm\alpha}$, $[e_\mu,[e_\nu,e_{-\sigma}]] = R^\alpha_{\mu,\nu,-\sigma}e_\alpha$, $[e_{-\mu},[e_{-\nu},e_\sigma]] = R^\alpha_{\mu,\nu,-\sigma}e_{-\alpha}$, where $R^\alpha_{\mu,\nu,-\sigma}$ are components of the curvature tensor defined at the initial point of the symmetric space N/H [20].

For the matrices from (23) we choose the following ansatz:

$$T_t = \sum_\alpha r^\alpha(e_\alpha - e_{-\alpha}) - i\sum_{\alpha,\beta}\Omega^{\alpha,\beta}[e_\alpha, e_{-\beta}], \quad T_u = A,$$

$$T_y = 0, \quad T_z = -i\sum_\alpha q^\alpha(e_\alpha + e_{-\alpha}), \tag{24}$$

where r^α, q^α and $\Omega^{\alpha,\beta}$ are arbitrary real-valued functions of ρ. Substituting (24) into eqs.(23), we obtain that

$$r^\alpha = \frac{d}{d\rho}q^\alpha + \frac{1}{\rho}q^\alpha, \quad \Omega^{\alpha,\beta} = q^\alpha q^\beta - \Omega^{\alpha,\beta}_o + 2\int\frac{d\rho}{\rho}q^\alpha q^\beta, \tag{25}$$

where $\Omega^{\alpha,\beta}_o = \Omega^{\beta,\alpha}_o = const$, and the functions q^α have to satisfy the deformed equations of motion in quartic potentials

$$\frac{d^2}{d\rho^2}q^\alpha - \sum_{\mu,\nu,\sigma} R^\alpha_{\mu,\nu,-\sigma}q^\mu q^\nu q^\sigma + \sum_{\mu,\nu,\sigma} R^\alpha_{\mu,\nu,-\sigma}\Omega^{\nu,\sigma}_o q^\mu =$$

$$= -\frac{d}{d\rho}(\frac{1}{\rho}q^{\alpha}) + 2\sum_{\mu,\nu,\sigma} R^{\alpha}_{\mu,\nu,-\sigma}q^{\mu}\int\frac{d\rho}{\rho}q^{\nu}q^{\sigma}. \quad (26)$$

The corresponding linear system is obtained by substituting (24) and (25) into (22).

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