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ON NONLINEAR MHD-STABILITY OF TOROIDAL MAGNETIZED PLASMAS

V.I. Ilgisonis, V.P. Pastukhov

Russian Research Centre "Kurchatov Institute"

123182 Moscow, Russia

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The variational approach to analyze the nonlinear magnetohydrodynamical (MHD) stability of ideal plasmas in magnetic fields of toroidal topology is proposed. The potential energy functional used as Lyapunov functional is expressed in terms of complete set of independent Lagrangian invariants, that allows to take strictly into account all the restrictions inherent in the varied functions due to MHD dynamic equations. It is shown that there are no physical MHD perturbations being able to grow if the linear MHD stability is provided.

Problem formulation

It is well known that the plasma stability problem does not be exhausted by the most conventional linear theory which cannot describe the perturbations growing slower than exponentially [1]. To analyze nonlinear plasma stability in frames of ideal magnetohydrodynamics (MHD), it is very attractive to use Lyapunov approach, choosing the plasma potential energy

$$W = \int d^3r \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) \tag{1}$$

as the Lyapunov functional to be varied, because its time derivative cannot be positive due to the total energy conservation resulted from ideal MHD equations (see, e.g.[2]). However, to avoid the possible narrowing the class of analyzed equilibria and the obtaining the more hard stability criterion than it is necessary, the certain restrictions on the varied functions have to be taken into account. Such the restrictions appear from the initial set of dynamic equations:

$$(\partial_t + \mathbf{V}\nabla)\rho + \rho \,\mathrm{div}\, \mathbf{V} = 0 \ , \tag{2}$$

$$(\partial_t + \mathbf{V}\nabla)\frac{p}{\rho^\gamma} = 0 \ , \tag{3}$$

$$\partial_t \mathbf{B} = \mathrm{curl}[\mathbf{V} \times \mathbf{B}] \ . \tag{4}$$

The values p and ρ denote the pressure and the density of plasma which moves with the velocity $\mathbf{V}$ in the magnetic field $\mathbf{B}$, γ is an adiabatic exponent. The quantities like ρ satisfying the continuity equation (2) are called Eulerian invariants, while the quantities like p/ρ^γ, which move together with plasma, are called Lagrangian invariants.

The above mentioned restrictions can be added to the functional (1) as a set of Eulerian invariants. Since the equations (2)-(4) (with the motion equation) are known to have the infinite set of invariants, the earlier attempts to modify the functional (1) by adding an incomplete set of invariants led to the narrowing the class of equilibria (see, e.g. [3-4]). Below the adequate procedure that allows to take strictly into account all the restrictions inherent in the varied functions under the integral (1) due to equations (2)-(4) is described.

Let us discuss the difference between our approach and linear theory more in detail. Linear MHD stability is also known to be analyzed by the similar energy principle [5], that declares that for linear instability it is necessary and sufficient to find a small trial plasma displacement ξ providing a quadratic form $\delta^2 W[\xi\ ,\xi]$ to be negative, when the second variation of W has to be calculated using the Eqs.(2)-(4) linearized over ξ. The point to apply a nonlinear theory is that even there is no such linearly unstable perturbations, nevertheless, there always are some "neutral" perturbations ξ_N, which do not change the above quadratic form:

$$\delta^2 W[\xi_N\ ,\xi_N] = 0\ .$$

Linear theory cannot say anything definite about the growth rate of such displacements. It can be determined only from the nonlinear analysis of the behavior of the potential energy in an equilibrium vicinity extended along those neutral displacements. Such an analysis has to be distinguished from investigations of nonlinear stages of linear instabilities.

Lyapunov functional construction

First of all some obvious properties of the invariants which result immediately from the equations (2)-(4) can be formulated:

a ratio of two Eulerian invariants is a Lagrangian one;

an arbitrary function of Lagrangian invariants is a Lagrangian invariant too;

if $\mathbf{B}$ satisfies the Eq.(4), the quantity $\mathbf{B}\nabla\alpha$ is an Eulerian invariant for any Lagrangian invariant α.

Let us introduce the system of independent coordinate $\{\mu,\nu,\lambda\}$, whose Jacobian $J = \nabla\mu[\nabla\nu \times \nabla\lambda] \neq 0$ everywhere in plasma for a given moment of time. It is always possible "to glue" those coordinates to the plasma, so they will obey the Eq.(3), that means they will be Lagrangian invariants. The Jacobian can be easily proved to be an Eulerian one [6], therefore, it cannot be equal to zero at any time. Using the Eqs.(2)-(3), the Eulerian invariant $p^{1/\gamma}$ can be constructed. As follows from the above mentioned invariant properties, it can be expressed through the Jacobian:

$$p = J^\gamma\, \Pi(\mu,\nu,\lambda)\ , \tag{5}$$

where function Π specifies the pressure equilibrium distribution.

An arbitrary magnetic field satisfying (4) can be expressed through Lagrangian coordinates as

$$\mathbf{B} = [\nabla\mu \times \nabla\nu]\mathcal{L} + [\nabla\nu \times \nabla\lambda]\mathcal{M} + [\nabla\lambda \times \nabla\mu]\mathcal{N}\ ,$$

where the functions $\mathcal{L},\mathcal{M},\mathcal{N}$ of Lagrangian coordinates have to obey the single condition $\partial_\lambda\mathcal{L} + \partial_\mu\mathcal{M} + \partial_\nu\mathcal{N} = 0$ to provide $\mathrm{div}\mathbf{B} = 0$. The closed magnetic field line system can be also described by this formula putting two functions (e.g., $\mathcal{M}$ and

$\mathcal{N}$) be equal to zero identically. Considering toroidal plasma only, let us choose the coordinate μ as a magnetic surface mark:

$$\mathbf{B}\nabla\mu = 0 \ , \tag{6}$$

that simply needs to put $\mathcal{M} = 0$ in the above general formula. Being satisfied at the initial moment, the Eq.(6) remains be valid at any time. Moreover, from our magnetic field representation follows immediately that the topology of magnetic field cannot be damaged during the time evolution (4). It should be mentioned that for any toroidal nested surfaces the coordinates $\{\mu, \ \nu, \ \lambda\}$ can be redefined to get the known Kruskal's flux representation for magnetic field [7]:

$$\mathbf{B} = [\nabla\tilde{\mu} \times (q(\tilde{\mu})\nabla\tilde{\nu} - \nabla\tilde{\lambda})] \ , \tag{7}$$

where $\{\tilde{\nu} \ , \tilde{\lambda}\}$ are analogs of poloidal and toroidal angles in tokamak respectively, and $q(\tilde{\mu})$ is the ratio of toroidal and poloidal fluxes ("safety factor"). That representation is also valid at any time (not only in equilibrium), and is often more suitable for use due to its simpler form (tildes will be omitted below).

Having substituted the explicit representation (5), (7) into the potential energy (1), we find that our functional (1) is only expressed through Lagrangian invariants μ, ν, λ without additional dependence on time. Now we may vary our functional over the coordinates μ, ν, λ independently, and there are no additional restrictions which we have to take into account.

First and second variations

Let's introduce the vector $\boldsymbol{\xi}$ by the relations:

$$\delta\mu = -\boldsymbol{\xi}\nabla\mu \ , \quad \delta\nu = -\boldsymbol{\xi}\nabla\nu \ , \quad \delta\lambda = -\boldsymbol{\xi}\nabla\lambda \ , \tag{8}$$

where δ denotes the corresponding variation. Having found the variations of the Jacobian $\delta J = -\mathrm{div}J\boldsymbol{\xi}$, we can easily obtain the variations of the pressure and magnetic field:

$$\delta p = -\boldsymbol{\xi}\nabla p - \gamma p \mathrm{div}\boldsymbol{\xi} \ , \quad \delta\mathbf{B} = \mathrm{curl}[\boldsymbol{\xi} \times \mathbf{B}] \ , \tag{9}$$

that coincides precisely with linear variations, if the formally introduced vector $\boldsymbol{\xi}$ (8) plays a role of a linear plasma displacement. The Eqs.(9) allow to derive the first and the second variation in the conventional form

$$\begin{aligned} \delta W &\approx \int d^3r\boldsymbol{\xi}(\nabla p + [\mathbf{B} \times \mathrm{curl}\mathbf{B}]) \ , \\ \delta^2 W &\approx \int d^3r((\delta\mathbf{B})^2 + \boldsymbol{\xi}[\delta\mathbf{B} \times \mathrm{curl}\mathbf{B}] - \delta p \mathrm{div}\boldsymbol{\xi}) \ , \end{aligned} \tag{10}$$

where δp and $\delta\mathbf{B}$ are given by Eq.(9). Here and below only the contributions of integrals over the plasma volume into potential energy variations will be considered for a brevity. The contribution of surface integrals results in the conventional conditions of plasma boundary equilibrium and transversality. Due to an arbitrariness of $\boldsymbol{\xi}$, the condition $\delta W = 0$ results in the general plasma equilibrium equation:

$$\nabla p + [\mathbf{B} \times \mathrm{curl}\mathbf{B}] = 0 \ . \tag{11}$$

It should be mentioned that in [3,4] only the narrow class of equilibria was obtained instead of (11) due to an incomplete accounting the consequences of Eqs.(2)-(4).

The second variation $\delta^2 W$ (10) coincides with a conventional energy principle [5] derived from the linear theory, that is obvious due to self-conjugation of quadratic functional. However, the principal difference between a linearization procedure and our approach leads to a different treatment of the results obtained.

Neutral displacements

Contrary to the linear theory our approach operates with the nonlinear energy functional (1) resulted from the precise nonlinear equation of motion. Furthermore, the proposed expressions (5), (7) for plasma pressure and magnetic field through the independent Lagrangian invariants represent all the relations between the variations of p and B due to nonlinear dynamic equations (2)-(4). It means that the higher order variations of the functional (1) can be also taken into account. Indeed, if $\delta^2 W \geq 0$ for any $\boldsymbol{\xi}$, it does not still provide a stability, because there always are some nontrivial neutral displacements $\boldsymbol{\xi}_N$ making $\delta^2 W|_{\boldsymbol{\xi}_N} = 0$. These neutral displacements satisfy the Euler equation

$$\nabla \delta p + [\delta \mathbf{B} \times \mathrm{curl}\mathbf{B}] + [\mathbf{B} \times \mathrm{curl}\delta\mathbf{B}] = 0 \ , \tag{12}$$

which was taken without fixing any norm of displacements considered. Displacements $\boldsymbol{\xi}_N$ satisfying (12) may be distinguished into three classes:

1. $\boldsymbol{\xi}_N \cdot \mathbf{n} \neq 0$; $\quad \boldsymbol{\xi}_N \cdot \mathbf{n}\Big|_S \neq 0 \quad (\mathbf{n} = \nabla\mu/|\nabla\mu|)$,

S denotes the boundary magnetic surface). Such displacements correspond to global equilibrium deformation (a case of near equilibria), and can be suppressed by external feedbacks, or something similar.

2. $\boldsymbol{\xi}_N \cdot \mathbf{n} \neq 0$; $\quad \boldsymbol{\xi}_N \cdot \mathbf{n}\Big|_S = 0$.

Such displacements don't change the certain equilibrium and correspond to the marginal stability situation, when the plasma confined is at the instability threshold. This situation is obvious to can be broken down by week changes of equilibrium parameters, therefore, it has to be out of our interest.

3. $\boldsymbol{\xi}_N \cdot \mathbf{n} = 0$ everywhere.

Such displacements can take place for an arbitrary equilibrium, therefore, they have to be analyzed at first.

Looking for an explicit form of those displacements $\boldsymbol{\xi}_N$, let us multiply (12) by B, we find $\delta\mathbf{B}\nabla p = -\mathbf{B}\nabla\delta p \Rightarrow \mathbf{B}\nabla \mathrm{div}\boldsymbol{\xi}_N = 0$. As $\boldsymbol{\xi}_N \ \mathbf{n} = 0$, hence, $\mathrm{div}\boldsymbol{\xi}_N = 0 \Rightarrow \delta p\Big|_{\boldsymbol{\xi}_N} = 0$ and $\delta\mathbf{B}\Big|_{\boldsymbol{\xi}_N} = 0$, that results in the following expression for $\boldsymbol{\xi}_N$:

$$\boldsymbol{\xi}_N = a(p) \ \mathrm{curl}\mathbf{B} + b(p)\mathbf{B} \ , \tag{13}$$

where a and b are arbitrary functions (although for rational magnetic surfaces the form (13) has to be modified, however, this fact cannot change the final conclusions given below).

Now, following our logic, we should investigate the question, are those neutral displacements able to give a contribution in the higher order variations of W?

Third and fourth variations

The reason to calculate higher order variations is that the neutral displacements may have larger amplitude then other ones. Therefore, if their contribution to the $\delta^3 W$ is not zero, it will be able to compete with $\delta^2 W$ calculated over the significant displacements $\boldsymbol{\xi}_* = \boldsymbol{\xi} - \boldsymbol{\xi}_N$ (different from neutral ones). The volume part of

$$\delta^3 W \approx \int d^3 r \Big\{ \delta^2 \boldsymbol{\xi}(\nabla p + [\mathbf{B} \times \mathrm{curl}\mathbf{B}]) + 2\delta\boldsymbol{\xi}\delta(\nabla p + [\mathbf{B} \times \mathrm{curl}\mathbf{B}]) + \boldsymbol{\xi}(\nabla \delta^2 p + [\delta^2 \mathbf{B} \times \mathrm{curl}\mathbf{B}] + 2[\delta\mathbf{B} \times \mathrm{curl}\delta\mathbf{B}] + [\mathbf{B} \times \mathrm{curl}\delta^2\mathbf{B}]) \Big\} \quad (14)$$

has to be calculated using that

$$\delta\boldsymbol{\xi} \cdot \nabla\alpha = \boldsymbol{\xi}\nabla(\boldsymbol{\xi}\nabla\alpha) \ , \quad (15)$$

where $\alpha = \{\mu, \nu, \lambda\}$ denotes any independent variable. Using the simple relations following from (15) one can find

$$\delta_N \boldsymbol{\xi}_N \cdot \nabla\mu = 0 \ , \qquad [\delta_N \boldsymbol{\xi}_N \, \mathbf{B}] = 0, \qquad \mathrm{curl}[\ \mathrm{curl}\mathbf{B} \times \boldsymbol{\xi}_N] = 0 \quad (16)$$

for $\delta_N \boldsymbol{\xi}_N : \delta_N \boldsymbol{\xi}_N \cdot \nabla\alpha = \boldsymbol{\xi}_N \nabla(\boldsymbol{\xi}_N \nabla\alpha)$. Substituting (16) into (14) we find that the biggest part of $\delta^3 W \sim ||\boldsymbol{\xi}_N||^3$ is canceled identically, and the term of order $\sim ||\boldsymbol{\xi}_*|| ||\boldsymbol{\xi}_N||^2$ is

$$\delta^3 W[\boldsymbol{\xi}_N, \boldsymbol{\xi}_N, \boldsymbol{\xi}_*] \approx \int 3\gamma p \ \mathrm{div}\boldsymbol{\xi}_* \ \mathrm{div}(\delta_N \boldsymbol{\xi}_N) \ d^3 r \, .$$

This term is able to compete with $\delta^2 W \sim ||\boldsymbol{\xi}_*||^2$ if

$$||\boldsymbol{\xi}_N|| \sim \sqrt{||\boldsymbol{\xi}_*||} \ . \quad (17)$$

The ordering (17) requires to calculate also the fourth variation term $\sim ||\boldsymbol{\xi}_N||^4$:

$$\delta^4 W[\boldsymbol{\xi}_N, \boldsymbol{\xi}_N, \boldsymbol{\xi}_N, \boldsymbol{\xi}_N] \approx \int 3\gamma p \ \mathrm{div}^2(\delta_N \boldsymbol{\xi}_N) \ d^3 r \, .$$

Summing all the terms of the same order, we find the change of Lyapunov functional near equilibrium as

$$\Delta W = \frac{1}{2}\ \delta^2 W + \frac{1}{6}\ \delta^3 W + \frac{1}{24}\ \delta^4 W \approx \approx \frac{1}{2} \int_{pl} \Big(\delta\mathbf{B}^2 + \boldsymbol{\xi}[\delta\mathbf{B} \times \mathrm{curl}\mathbf{B}] + \boldsymbol{\xi}\nabla p \ \mathrm{div}\boldsymbol{\xi} + \gamma p \ \mathrm{div}^2(\boldsymbol{\xi} + \frac{1}{2}\ \delta_N \boldsymbol{\xi}_N) \Big) d^3 r . \quad (18)$$

This formula differs from the expression for $\frac{1}{2}\ \delta^2 W$ by a correction in the last compressible term. Due to (16) that correction can be canceled by redefining $\xi_{\parallel}$ without any change of other terms of (18).

Finite neutral displacements

The result of the previous section can be interpreted as a next order correction of the neutral displacement. It means that real neutral displacement has a form:

$$\xi_N = \xi_N^0 + \xi_N^1 + \xi_N^2 + \cdots \ , \tag{19}$$

where ξ_N^0 is defined by (13), $\xi_N^1 = -\frac{1}{2}\delta_N \xi_N^0$, etc. It is just a displacement that doesn't change pressure and magnetic field with counting terms of order (17). The counting higher order variations results effectively in the similar (18) corrections to the quadratic functional $\frac{1}{2}\ \delta^2 W$, corresponding to the next order term in expansion (19). Looking for finite neutral displacements instead of infinitesimal expansion (19), one can find the following coordinate transform

$$\begin{aligned} \alpha = \{\mu, \nu, \lambda\} \quad &\rightarrow \quad \tilde{\alpha} = \{\tilde{\mu}, \tilde{\nu}, \tilde{\lambda}\} : \\ \mu \quad &= \quad \tilde{\mu} \ ; \\ F(\mu, \nu, \lambda) \quad &= \quad F(\tilde{\mu}, \tilde{\nu}, \tilde{\lambda}) \ , \text{where} \\ F(\mu, \nu, \lambda) \quad &= \quad F_1(\mu) \int_0^{\nu} \frac{d\zeta}{J_0(\mu, \zeta, \lambda + q(\zeta - \nu))} + F_2(\mu, \lambda - q\nu) \ , \end{aligned} \tag{20}$$

and F_1, F_2 - arbitrary functions (F_2 has only to provide the partial derivatives of F be physical functions, 2π-periodic of angular-like variables ν, λ; such a function F_2 can always be found), J_0 - Jacobian in an equilibrium state, which determines the function Π in (5) as a ratio $p_0(\mu)/J_0^{\gamma}(\mu, \nu, \lambda)$. It can be easily proved that the transform (20) provides the following relations between initial ($J = \nabla\mu[\nabla\nu \times \nabla\lambda]$) and final ($\tilde{J} = \nabla\tilde{\mu}[\nabla\tilde{\nu} \times \nabla\tilde{\lambda}]$) Jacobians

$$J = \tilde{J} \frac{J_0(\mu, \nu, \lambda)}{J_0(\tilde{\mu}, \tilde{\nu}, \tilde{\lambda})}$$

and $\tilde{\mathbf{B}} = \mathbf{B}$, $\tilde{p} = p$ due to the definitions (5), (7). Hence the transform (20) does describe the nontrivial finite neutral displacements whose infinitesimal expansion (19) has been found earlier.

Discussion

The explicit representation of pressure (5) and magnetic field (7) by a set of independent Lagrangian invariants allowed us to vary plasma potential energy strictly taking into account all the relations followed from MHD equations. That variational procedure resulted in general equilibria and stability criterion looking the same as in linear stability theory. For *any* equilibrium there are infinitesimal neutral displacements (13), which do not change potential energy to the second order (10). These infinitesimal neutral perturbations are extended to the finite displacements. From the mathematical point of view, the presence of such finite neutral displacements should be considered as instability, because at least one of coordinates $\{\nu, \lambda\}$ is able to grow infinitely. However, slow motion along such neutral directions changes no macroscopic plasma parameters. There is no coupling those neutral displacements of arbitrary amplitude with any others. Therefore, from a physical point of view they correspond to coordinate relabeling and cannot be considered as a real instability to be interested. In other words, contrary to conclusions of previous papers (e.g., [3,4]) there is no nonlinear MHD instability for *any* static equilibria when the linear plasma stability is provided.

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