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SOLUTION OF SELF-DUALITY EQUATION IN QUANTUM-GROUP GAUGE THEORY

B.M.Zupnik[1)]

Bogoliubov Laboratory of Theoretical Physics , JINR
141980 Dubna, Moscow Region, Russia

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We discuss a gauge theory for the quantum group $SU_q(2) \times U(1)$ on the quantum Euclidean space. This theory contains three physical gauge fields and one $U(1)$-gauge field with a zero field strength. We construct the quantum-group self-duality equation (QGSDE) in terms of differential forms and with the help of the field-strength decomposition. A deformed analogue of the BPST-instanton solution is obtained. We consider the possibility of a harmonic (twistor) interpretation of QGSDE in terms of quantum harmonics.

An attractive idea of quantum deformations for the gauge theories has been considered in the framework of different approaches [1–3]. Formally one can discuss independent deformations of basic spaces and gauge groups and possible correlations between these deformations. We shall here consider a gauge theory with identical one-parameter deformations of the 4-dimensional Euclidean space and the gauge group $SU(2)$. A consistent formulation of the gauge theory for the semisimple quantum group $SU_q(N)$ is unknown to us, so we shall deal with the quantum group $U_q(2) = SU_q(2) \times U(1)$. It will be shown that the $U(1)$-gauge field can be treated as a field with a zero field strength.

Consider the standard relations between elements T^i_k of the quantum $U_q(2)$-matrix [4]

$$R\,T\,T' = T\,T'\,R \quad \Leftrightarrow \quad R^{ik}_{lm} T^l_j T^m_n = T^i_l T^k_m R^{lm}_{jn}, \tag{1}$$

where I is a unity matrix , R is the constant symmetric matrix with components $R^{ik}_{lm}(q)$ $(i, k, l, m = 1,\ 2)$ and q is a real deformation parameter. We use the R-matrix method in the condensed notations of ref. [3] (see also [4,5]). A translation of matrix formulae to the usual index notation can be fulfilled with the help of the following substitution:

$$R \Rightarrow R^{ik}_{lm},\quad T \Rightarrow (T\otimes I)^{lm}_{nj} = T^l_n \delta^m_j,\quad T' \Rightarrow (I \otimes T)^{nj}_{ps} = \delta^n_p T^j_s. \tag{2}$$

The parameters $q(ik)$ $(q(12) = q,\ q(21) = q^{-1},\ q(11) = q(22) = 1)$ define a q-deformation of the ε-symbol $\varepsilon_{ik}(q) = \sqrt{q(ik)}\varepsilon_{ik}$ where ε_{ik} is the ordinary anti-symmetric symbol.

The R-matrix can be written in terms of projection operators $P^{(\pm)}$: $R = qP^{(+)} - q^{-1}P^{(-)}$, $P^{(+)} + P^{(-)} = I$. The operator $P^{(-)}$ for $U_q(2)$ is proportional to the product of two $\varepsilon(q)$-symbols:

$$[P^{(-)}]^{ik}_{lm} = -\frac{q}{1+q^2}\varepsilon^{ki}(q)\varepsilon_{ml}(q), \qquad \varepsilon^{ki}(q)\varepsilon_{il}(q) = \delta^k_l. \tag{3}$$

Here the basic identity for $\varepsilon(q)$-symbols with upper indices is written also.

[1)]e-mail: zupnik@thsun1.jinr.dubna.su

We shall use the covariant relation for the quantum determinant $D(T)$ of the $U_q(2)$-matrix

$$\varepsilon_{ml}(q)T^l_i\, T^m_k = \varepsilon_{ki}(q)\; D(T). \tag{4}$$

A covariant expression for the inverse quantum matrix $S(T) = T^{-1}$ can be obtained from this equation.

The $SU_q(2)$-metric $\mathcal{D}(q)$ determines the matrix product of transposed q-matrices[4]

$$T^l_i \mathcal{D}^m_l(q) S(T)^k_m = \mathcal{D}^k_i(q) = -\varepsilon_{mi}(q)\varepsilon^{mk}(q)\,. \tag{5}$$

The unitarity condition for the matrix T can be formulated with the help of involution [4] $\overline{T^i_k} = S^k_i$.

Let us consider the bicovariant differential calculus on the $U_q(2)$ group [5-7]

$$T\, dT' = R\, dT\, T'\, R, \qquad D(T)\, dT = q^2 dT\, D(T). \tag{6}$$

Note that the condition $D(T) = 1$ is inconsistent in the framework of this calculus. Consider the relations for the right-invariant differential forms $\omega = dT\, S$ [3]

$$\omega R \omega + R\omega R\omega R = 0\,, \quad T\,\omega' = R\,\omega\, R\, T. \tag{7}$$

The quantum trace ξ of the form ω plays an important role in this calculus

$$\xi(T) = \mathcal{D}^k_i(q)\omega^i_k(T) \neq 0, \quad \xi^2 = 0, \quad d\xi = 0. \tag{8}$$

$$dT = \omega T = (q^3 - q)^{-1}[T, \xi], \quad q dD(T) = \xi D(T), \qquad d\omega = \omega^2 = (q - q^3)^{-1}\{\xi, \omega\}\,. \tag{9}$$

Note that the basic relations of the bicovariant calculus on $GL_q(2)$ and $U_q(2)$ have been analysed in details [5–7]. We shall call this calculus as BC-calculus.

The BC-calculus makes the basis for consistent formulation of quantum-group gauge theory in the framework of noncommutative algebra of differential complexes [2-3]. Consider formally the quantum group gauge matrix $T^a_b(x)$ defined on some basic space. Suppose that eqs. (4)–(6) satisfy locally for each "point" x. Then one can try to construct the $U_q(2)$-connection 1-form $A^a_b(x)$ which obeys the simplest commutation relation

$$(A\ R\ A + R\ A\ R\ A\ R\ )^{ab}_{cd} = A^a_e R^{eb}_{gd} A^g_c + R^{ab}_{ef} A^e_g R^{gf}_{hn} A^h_m R^{mn}_{cd} = 0. \tag{10}$$

These relations generalize the anticommutativity conditions for components of the classical connection form. Note that the general relation for A contains a nontrivial right-hand side [3].

Coaction of the gauge quantum group $U_q(2)$ has the following standard form:

$$A \to T(x)\, A\, S(T) + dT(x)\, S(T) = T\, A\, S + \omega(T), \qquad \alpha = \mathrm{Tr}_q A \to \alpha + \xi(T). \tag{11}$$

The restriction $\alpha = 0$ is inconsistent with (10), but we can use the gauge-covariant relations $\alpha^2 = 0$ and $\mathrm{Tr}_q A^2 = 0$.

It should be stressed that we can choose the zero field-strength condition $d\alpha = 0$ for the $U(1)$-gauge field [2)]. This constraint is gauge invariant and consistent with (10). The deformed pure gauge field α can be decoupled from the set of physical fields in the limit $q = 1$. We shall consider further the $U_q(2)$-gauge theory with three "physical" gauge fields and one "zero-mode" $U(1)$ field. The curvature 2-form $F = dA - A^2$ is q-traceless for this model.

Quantum deformations of Minkowski and Euclidean 4-dimensional spaces have been considered in refs. [8, 9]. We shall treat the coordinates x^i_α of q-deformed Euclidean space $E_q(4)$ as generators of a noncommutative algebra $(Rx\,x' = x\,x'R)$ covariant under the coaction of the quantum group $G_q(4) = SU^L_q(2) \times SU^R_q(2)$. The q-deformed central Euclidean interval τ can be constructed by analogy with the quantum determinant

$$\tau = x^1_1 x^2_2 - q x^1_2 x^2_1 = -\frac{q}{1+q^2}\varepsilon^{\beta\alpha}(q)\varepsilon_{ki}(q)x^i_\alpha x^k_\beta. \tag{12}$$

We do not consider the quantum group structure on $E_q(4)$. It is convenient to use the following $E_q(4)$ involution

$$\overline{x^i_\alpha} = \varepsilon_{ik}(q)x^k_\beta \varepsilon^{\beta\alpha}(q) = \tau S^\alpha_i(x), \tag{13}$$

$$\overline{\tau} = \tau\,, \quad \overline{\overline{x^i_\alpha}} = x^i_\alpha, \tag{14}$$

where $S(x)$ is an inverse matrix for the matrix x.

We shall use an analogue of the bicovariant $U_q(2)$-calculus for studying differential complexes on $E_q(4)$. The commutation relations between matrices x and dx can be obtained from eqs. (6 - 9) by formal substitution $T \to x$. One can obtain , for example

$$x^i_\alpha dx^k_\beta = R^{ik}_{lm} dx^l_\gamma x^m_\rho R^{\gamma\rho}_{\alpha\beta}, \tag{15}$$
$$P^{(+)}dx dx' P^{(+)} = 0 = P^{(-)}dx dx' P^{(-)}.$$

The basic decomposition of 2-forms on $E_q(4)$ has the following form

$$dx^i_\alpha dx^k_\beta = [P^{(-)}dx dx' + dx dx' P^{(-)}]^{ik}_{\alpha\beta} = \varepsilon^{ki}(q)d^2x_{\alpha\beta} + \varepsilon_{\beta\alpha}(q)d^2x^{ik}, \tag{16}$$

where eq. (3) for $P^{(-)}$ is used. By analogy with the classical case we can treat two terms of this decomposition as self-dual and anti-self-dual 2-forms under the action of a duality operator $*$.

Consider the right-invariant 1-forms on $E_q(4)$

$$\omega^i_k(x) = [dx S(x)]^i_k = dx^i_\alpha S^\alpha_k, \quad dx = \omega x \tag{17}$$

where $S(x)$ is the inverse matrix for x defined by eq. (13). It is convenient to rewrite the decomposition of 2-forms in terms of the right-invariant self-dual and anti-self-dual forms

$$dx dx' = \omega x \omega' x' = \omega R \omega R x x', \tag{18}$$

[2)] This condition is consistent also for the case of $GL_q(N)$ group.

$$P^{(-)}dx\,dx' = qP^{(-)}\omega R\omega xx'P^{(+)} = P^{(-)}\Omega_S P^{(+)}xx', \tag{19}$$

$$dx\,dx'P^{(-)} = \Omega_A P^{(-)}xx', \tag{20}$$

$$\Omega_S = *\Omega_S = q^4\omega^2 + q\omega\xi, \tag{21}$$

$$\Omega_A = -(*\Omega_A) = q^{-1}\omega\xi - \omega^2. \tag{22}$$

Here the comutation relations of BC-calculus on $E_q(4)$ and properties of the $P^{\pm}$-operators were used. It should be stressed that the condensed notations simplify significantly these calculations.

Let us introduce the simple ansatz for quantum $U_q(2)$ anti-self-dual gauge fields

$$A^a_b = dx^i_\alpha A^{\alpha a}_{ib}(x) = \omega^a_b(x)f(\tau), \qquad A^{\alpha a}_{ib}(x) = \delta^a_i S^\alpha_b(x)f(\tau), \tag{23}$$

where $f(\tau)$ is a function of q-interval (12). Note that this ansatz is a partial case of more general construction of the differential complex on $GL_q(2)$ [2,3]. Addition of the term $\xi(x)g(\tau)$ results in a relation for the connection A more complicated than (10) .

Consider the q-traceless curvature form for the connection (23) which can be calculated in the framework of the BC-calculus on $E_q(4)$

$$F = \omega^2 f(\tau)[1 - f(q^2\tau)] + (q^3 - q)^{-1}\omega\xi[f(\tau) - f(q^2\tau)]. \tag{24}$$

An appearance of the finite translation $f(q^2\tau)$ is a general feature of the calculus on the quantum space.

The anti-self-duality equation for our ansatz is equivalent to the nonlinear finite-difference equation

$$*F = -F \;\Rightarrow\; F \sim \Omega_A f(\tau)[1 - f(q^2\tau)], \tag{25}$$

$$f(\tau) - f(q^2\tau) = (1 - q^2)f(\tau)[1 - f(q^2\tau)], \tag{26}$$

where Ω_A is the anti-self-dual 2-form (22).

This equation has a simple solution analogous to the classical BPST-solution

$$f(\tau) = \frac{\tau}{a + \tau}, \tag{27}$$

where a is an arbitrary "constant" that can be treated as a central periodical function: $a(\tau) = a(q^2\tau)$. Note that our solution for the connection A contains the parameter q only through definitions of $\omega(x)$ and τ, however, the corresponding curvature has more explicit q-dependence (24).

It is easy to obtain the 5-parameter solution via substitution [3)] $x^i_\alpha \to \hat{x}^i_\alpha = x^i_\alpha + c^i_\alpha$ in eqs. (23), (27). Stress that our anti-self-dual solution is a function on the braided algebra with the noncommuting generators x, dx, a and c.

$$R\,\hat{x}\,\hat{x}' = \hat{x}\,\hat{x}'\,R, \quad R\,c\,c' = c\,c'\,R, \quad c\,x' = R\,x\,c'\,R, \tag{28}$$

$$c\,dx' = R\,dx\,c'\,R, \quad [\hat{x}, \tau(\hat{x})] = 0, \tag{29}$$

$$d\hat{x} = dx, \quad dc = 0,$$

$$\tau(\hat{x})dx = q^2 dx\tau(\hat{x}).$$

[3)] The addition of q-matrices was considered early by V.Jain, O.Ogievetsky and S.Majid.

The QGSD-equation can be written in terms of the field strength

$$F = dx^i_\alpha dx^k_\beta F^{\beta\alpha}_{ki}(x)\,, \quad F^{\beta\alpha}_{ki} = \varepsilon_{ki}(q)F^{\beta\alpha}. \tag{30}$$

Introduce the additional noncommutative harmonic (twistor) variables u^i_+ satisfying the relations $\varepsilon_{ki}(q)u^i_+u^k_+ = 0$. One can obtain the integrability condition multiplying the QGSD-equation by the product $u^i_+u^k_+$. The analogous integrability conditions are the basis of the harmonic (twistor) approach to the classical self-duality equation [10, 11]. We considered the deformed harmonic formalism of QGSDE in ref. [12].

It seems very interesting to study reductions of QGSDE to lower dimensions and to search a more general deformation scheme for the self-duality equation.

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