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STABILITY OF TWO- AND THREE-DIMENSIONAL OPTICAL SOLITONS IN A MEDIA WITH QUADRATIC NONLINEARITY

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It is demonstrated that solitons existing due to mutual trapping of the fundamental and second-harmonic waves in a dispersive media with cascaded second-order nonlinearity are stable. It is proved that solitons realize the minimum of the Hamiltonian for the fixed Manly–Row integral.

Recent progress in studying materials with high second-order nonlinearities (see, e.g., [1] and references therein) stimulates interest to the problem of nonlinear waves propagation in such media. Dielectric materials with quadratic nonlinearity (the so-called χ^2 materials) provide one of the fastest electronic nonlinearities available to date. A possibility to generate in such materials a large intensity-dependent phase shift is of interest for future all-optical devices [2]. Experimental observation of the cascading process has been reported recently in [3]. In fact, the existence of solitons due to mutual trapping of the fundamental and second-harminic waves in the media with quadratic nonlinearities has been predicted in [4]. Structure and dynamics of temporal and spatial solitons in the χ^2 materials have been investigated in [4–8]. These studies have revealed the existence of various types of bright and dark solitons [6–8]. The central question for the physical relevance of these solitons is their stability.

In this Letter we study stability of multidimensional solitons that exist due to parametric interactions between the fundamental and second harmonics in quadratic nonlinear media. We demonstrate that solitons are stable because they realize a minimum of the Hamiltonian for fixed Manly–Row integral for any dimension of the problem.

Spatiotemporal evolution of the dimensionless slowly varying envelopes of the fundamental U and second harmonic V waves is governed by the following basic system written in the Hamiltonian form

$$i\frac{\partial U}{\partial \xi} = -\Delta U - U^* V = \frac{\delta H}{\delta U^*},$$
$$i\frac{\partial V}{\partial \xi} = -\frac{1}{2}\Delta V - \frac{1}{2}U^2 = \frac{\delta H}{\delta V^*}, \qquad (1)$$

here we use notation of Ref. [6], ξ is the normalized propagation distance, Δ is Laplasian. The Hamiltonian reads $H = I_1 - I_2$ where

$$I_1 = \int [|\vec{\nabla} U|^2 + \frac{1}{2}|\vec{\nabla} V|^2] dr, \quad I_2 = \frac{1}{2}\int [U^{*2} V + U^2 V^*] dr.$$

Besides the Hamiltonian, the Manly-Row integral $P = \int [|U|^2 + 2|V|^2]dr$ is a consreved quantity.

We study stability of the two- and three-dimensional localized symmetrical stationary solutions of Eqs. (1) of the form $U = \exp(i\lambda\xi)f(r)$, $V = \exp(i\lambda\xi)g(r)$.

The profile of solitons is given by the nonlinear eigenvalues problem for λ, f and g.

$$\begin{aligned} &-\lambda f + \Delta f + fg = 0, \\ &-4\lambda g + \Delta g + f^2 = 0. \end{aligned} \tag{2}$$

This system of equations can be rewritten in a variational form

$$\delta(H + \lambda P) = 0. \tag{3}$$

This means that solitons realize extremum of the Hamiltonian for the fixed P. We will show that a ground symmetrical solutions realize a minimum of the H. From Eq. (3) we can directly express the Hamiltonian in terms of P and λ on the soliton solution. Indeed, consider trial functions for the variational problem (3) in the form $f_1 = \alpha f_{sol}$ and $g_1 = \alpha g_{sol}$, where f_{sol}, g_{sol} stand for ground solutions of Eq. (2). Varying α near 1 we find that

$$\frac{\partial}{\partial\alpha}|_{\alpha=1}(H + \lambda P) = 2I_{1sol} - 3I_{2sol} + 2\lambda P_{sol} = 0. \tag{4}$$

The similar procedure with trial functions of the form $f_2 = f_{sol}(\beta r)$, $g_2 = g_{sol}(\beta r)$ give the realation

$$\frac{\partial}{\partial\beta}|_{\beta=1}(H + \lambda P) = (d-2)I_{1sol} - dI_{2sol} + d\lambda P_{sol} = 0, \tag{5}$$

where d is a dimension of the problem. Straightforward algebraic manipulations yield

$$H_{sol} = -\frac{4-d}{6-d}\lambda P_{sol}. \tag{6}$$

Using simple scaling we can present ground solutions in the form $f_{sol}(r,\lambda) = \lambda f_0(\sqrt{\lambda}r)$, $g_{sol}(r,\lambda) = \lambda g_0(\sqrt{\lambda}r)$. Here we introduce f_0 and g_0 as a ground solution of Eq. (2) with $\lambda = 1$. Now P_{sol} can be expressed in terms of $P_0 = P_{sol}[f_0, g_0]$:

$$P_{sol} = \lambda^{\frac{4-d}{2}} P_0. \tag{7}$$

The Hamiltonian on the ground solution can be written in the following form

$$H_{sol} = -\frac{4-d}{6-d}\left(\frac{P_{sol}}{P_0}\right)^{\frac{2}{4-d}} P_{sol}. \tag{8}$$

To demonstrate that the ground soliton solution realizes minimum of the Hamiltonian for the fixed P, we need to prove some interpolation estimate for I_2 through I_1 and P. Consider minimization problem for the functional $J[f,g] = P^{(6-d)/4} I_1^{d/4} / I_3$. It can be shown that a minimum is attained on the ground symmetrical soliton solution of the Eq. (2). In the proof we follow the procedure used in [9] (see for details [10]). Functional J is invariant under transformation $\tilde{f} = \nu f(\mu r), \tilde{g} = \nu g(\mu r)$. Thus, by scaling we can take $I_1 = \frac{d}{6-d}\lambda P$ and $I_3 = \frac{4}{6-d}\lambda P$. Computing the Euler–Lagrange equation for J leads to the Eq. (2). Indeed,

$$\frac{6-d}{4}\frac{1}{P}f - \frac{d}{4}\frac{1}{I_1}\Delta f - \frac{1}{I_3}f^* g = 0 \tag{9}$$

and

$$4\frac{6-d}{4}\frac{1}{P}g - \frac{d}{4}\frac{1}{I_1}\Delta g - \frac{1}{I_3}f^2 = 0 \ . \tag{10}$$

Using simple scaling transformation it is easy to obtain Eq. (2) from these equations. Minimum is attained at a functions f and g which are positive and functions of r alone. The compactness lemma providing that such a solution exists has been proved in [9].

Thus, we find that the minimum of J is attained at the ground soliton solution and can be calculated as

$$\min(J) = C_0 = (\frac{d}{6-d})^{d/4}(\frac{6-d}{4})P_0^{1/2}. \tag{11}$$

From this one can obtain an interpolation estimate with the "best constant" C_0: $I_3 \leq C_0 P^{(6-d)/4} I_1^{d/4}$. Substituting this estimate into the Hamiltonian, we come to the estimate of the H from below for the fixed P

$$H = I_1 - I_3 \geq I_1 - C_0 P^{\frac{6-d}{4}} I_1^{\frac{d}{4}} \geq P^{\frac{6-d}{4-d}} C_0^{-\frac{4}{4-d}}[(d/4)^{4/(4-d)} - (d/4)^{d/(4-d)}] =$$

$$= -\frac{4-d}{6-d}P^{\frac{6-d}{4-d}}P_0^{\frac{2}{d-4}} = H_{sol}. \tag{12}$$

Thus, the ground symmetrical soliton solution of Eq. (2) realizes minimum of the Hamiltonian. Now it is easy to see that a functional $L = H - H_{sol}$ satisfies all requirements to the Lyapunov function and the solitons are stable due to Lyapunov's theorem.

In conclusion, we have shown that solitons due to mutual trapping of the fundamental and second-harminic waves propagating in a medium with quadratic nonlinearities are stable. The approach and results obtained can be easily applied to other models of different physical context where parametric wave interactions are generated by quadratic nonlinearities.

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