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NEW CORRECTIONS TO HYPERFINE SPLITTING AND LAMB SHIFT AND THE VALUE OF THE RYDBERG CONSTANT

M.I.Eides[1), *V.A.Shelyuto**

Petersburg Nuclear Physics Institute,
188350 Gatchina, St.Petersburg, Russia

**D.I.Mendeleev Institute of Metrology,*
198005 St.Petersburg, Russia

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Corrections to hyperfine splitting and Lamb shift of order $\alpha^2(Z\alpha)^5$ induced by the diagrams with radiative photon insertions in the electron line are calculated in the Fried-Yennie gauge. These contributions are equal to $-7.725(1)\alpha^2(Z\alpha)^5/\pi n^3(m_r/m)^3 m$ and $-0.6726(4)\alpha^2(Z\alpha)/\pi n^3 E_F$ for the Lamb shift and hyperfine splitting, respectively. Phenomenological implications of these results are discussed.

Theoretical work on the high order corrections to hyperfine splitting (HFS) and Lamb shift concentrated recently on contributions of order $\alpha^2(Z\alpha)^5$. There are six gauge invariant sets of diagrams, which produce corrections of order $\alpha^2(Z\alpha)^5$ [1, 2]. All contributions induced by the diagrams, containing closed electron loops, were obtained recently in papers [1-5] for the case of hyperfine splitting and in papers [2, 6-9] for the case of the Lamb shift.

We report in this paper on the results of our calculation of the contributions of order $\alpha^2(Z\alpha)^5$ to HFS and Lamb shift induced by the gauge invariant set of nineteen topologically different diagrams with insertions of two radiative photons (see Table I) 2).

For the total correction of order $\alpha^2(Z\alpha)^5$ to the HFS and the Lamb shift produced by all nineteen diagrams with radiative photon insertions in the electron line we obtain

$$\Delta E^{(r)}_{HFS} = -0.6726(4)\frac{\alpha^2(Z\alpha)}{\pi n^3}\ E_F, \tag{1}$$

and

$$\Delta E^{(r)}_{L} = -7.725(1)\frac{\alpha^2(Z\alpha)^5}{\pi n^3}\ (\frac{m_r}{m})^3\ m. \tag{2}$$

While this work was in progress two other papers were published where the contributions of the same nineteen diagrams to HFS [5] and the Lamb shift [12] have been calculated. Despite the great differences in the approaches used in the present work and in [5-12] numerical factors in Eq. (1) and Eq. (2) are compatible with $-0.63(4)$ in [5] and with $-7.61(16)$ in [12], respectively. Our numbers are about two orders of magnitude more precise and further improvement of accuracy may be achieved. The reason for this increased accuracy is the use of the FY gauge, where one can avoid the extrapolation in the photon mass.

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1) e-mail: eides@lnpi.spb.su

2) We have calculated the contributions induced by first nine diagrams earlier [10,11]. Detailed account of our calculations will be presented in a separate publication.

TABLE I

Corrections to HFS and Lamb shift

Diagram	HFS $\frac{\alpha^2(Z\alpha)}{\pi n^3}E_F$	Lamb shift $\frac{\alpha^2(Z\alpha)^5}{\pi n^3}(\frac{m_r}{m})^3 m$
a	9/4	0
b	−6.65997(1)	2.9551(1)
c	3.93208(1)	−2.2231(1)
d	-3.903368(79)	−5.238023(56)
e	4.566710(24)	5.056278(81)
f	−3.404163(22)	−1.016145(21)
g	2.684706(26)	−0.1460233(52)
h	33/16	153/80
i	0.054645(46)	−5.51658(54)
j	−7.14937(16)	−7.76813(18)
k	1.465834(20)	1.959589(33)
l	−1.983298(95)	1.74815(38)
m	3.16956(16)	1.87540(17)
n	−3.59566(14)	−1.30584(18)
o	1.804775(46)	−12.06751(47)
p	3.50608(16)	6.13748(30)
q	−0.80380(15)	−7.52525(74)
r	1.05298(18)	14.36733(44)
s	0.277203(27)	−0.930268(72)
Total	−0.6726(4)	−7.725(1)

Numerically the correction to muonium HFS in the ground state produced by the diagrams under consideration is equal to

$$\Delta E^{(r)}_{HFS} = -0.3710(2)\text{kHz} \tag{3}$$

and the total contribution of order $\alpha^2(Z\alpha)E_F$ is given by

$$\Delta E_{HFS} = 0.4256(2)\text{kHz}. \tag{4}$$

Collecting all theoretical contributions to HFS (see, e.g., reviews in [5, 13]) and using for calculation the value of α from [14] one may obtain the theoretical value for the muonium HFS in the ground state

$$\Delta E_{HFS} = 4463302.55(0.18)(0.18)(1.33)\text{kHz}, \tag{5}$$

where the first error reflects the uncertainty of the fine structure constant, the second is induced by the uncertainty of the contribution of order $\alpha(Z\alpha)^2E_F$, while the third is determined by the experimental error in measuring electron-muon mass ratio m/M.

The agreement between theory and experiment [15] is excellent. Phenomenological situation and the influence of the result in Eq. (1) on the values of the electron-muon mass ratio and the fine structure constant is discussed in great detail [5,13].

The case of Lamb shift deserves more comments. Numerically the corrections to the $1S$ and $2S$ Lamb shifts produced by the nineteen diagrams under consideration are equal to

$$\Delta E^{(r)}_L(1S) = -334.\ 25\ (4)\ \text{kHz}, \tag{6}$$

$$\Delta E_L^{(r)}(2S) = -41.\ 781\ (5)\ \text{kHz},$$

while respective total contributions of order $\alpha^2(Z\alpha)^5 m$ are given by

$$\Delta E_L(1S) = -296.\ 94\ (4)\ \text{kHz}, \tag{7}$$

$$\Delta E_L(2S) = -37.\ 117\ (5)\ \text{kHz}.$$

Consider now briefly the current status of the Lamb shift theory. Theoretical predictions presented below are obtained with the help of the expressions for the Lamb shift contributions as collected in the reviews [16,17], amended, besides corrections obtained above and in [12], with some other recent results [18-23].

The accuracy of calculations of the Lamb shift intervals is limited by the magnitude of the yet uncalculated contributions of orders $(Z\alpha)^6(m/M)m$, $\alpha^3(Z\alpha)^4 m$ and $\alpha^2(Z\alpha)^6 m$. Our estimate of the theoretical uncertainty of the expression for the Lamb shift is about 28 kHz for the $1S$-state and about 4 kHz for the $2S$-state.

The other limit on the accuracy of the theoretical calculation of the Lamb shift is put by the experimantal value of the proton rms charge radius. There are two contradictory experimental results for this radius [24,25]. The accuracy of the proton rms charge radius claimed by the authors of [24,25] produces uncertainty about 32 kHz for the $1S$-state and about 4 kHz for the $2S$-state.

TABLE II

$2S_{1/2} - 2P_{1/2}$ **Lamb Shift**

ΔE (kHz)	
1 057 845 (9)	Experimental result, Ref. [26].
1 057 857. 6 (2.1)	Experimental result, Ref. [27,28].
1 057 839 (12)	Experimental result, Ref. [29].
1 057 810 (4) (4)	Theory, this work, and $r_p = 0.805\ (11)$ fm, Ref. [24].
1 057 829 (4) (4)	Theory, this work, and $r_p = 0.862\ (12)$ fm, Ref. [25].
1 057 854 (16)	Self-consistent 1S, Ref. [30].
1 057 835 (15)	Self-consistent 1S, Refs. [31,32].
1 057 847 (13)	Self-consistent 1S, Ref. [33].

Experimental data for the $2S_{1/2} - 2P_{1/2}$ Lamb shift and the results of our theoretical calculations are presented in Table II. The first error of the theoretical values in Table II is determined by the yet uncalculated contributions to the Lamb shift and the second reflects the experimental uncertainty of the proton rms charge radius. There are two immediate conclusions of the data in Table II. First, as already mentioned in [12], the results of the proton rms radius measurement in [24] should be in error since respective value of the proton charge radius is clearly inconsistent with all results of the Lamb shift measurements. Second, we have to reject either the result of the most precise measurement of the $2S_{1/2} - 2P_{1/2}$ splitting, or the experimental value of the proton charge radius as measured in [25] since the Lamb shift value in [27] contradicts theoretical value calculated employing the rms radius in [25] by more than five standard deviations. Results of two other measurements of the classic Lamb shift are compatible with the theory, so we will below accept the value of the proton charge radius as obtained in [25].

Unbiased extraction of the $1S$ Lamb shift from the experimental data is still a problem. The standard approach consists in adopting one or other $2S_{1/2} - 2P_{1/2}$ experimental result and extracting with its help the value of the $1S$ Lamb shift from the experimental data. All experimental values in Table III for the $1S$ Lamb shift are obtained in this manner with the help of the experimental values in [26] or in [29] for the classic Lamb shift. These values should be compared with our theoretical prediction also cited in the Table. The results of all experiments mentioned in Table III are pretty consistent and their agreement with the theoretical value is satisfactory.

One can extract self-consistent values of the $1S$ Lamb shift from the experimental data [30,31,33] unambiguously without reference to the $2S_{1/2} - 2P_{1/2}$ experimental results with the help of the theoretical relation between the $1S$ and $2S$ Lamb shifts

$$8E_L(2S) - E_L(1S) = \Delta, \tag{8}$$

where $\Delta = 187\ 234\ (7)$ kHz. The difference $8E_L(2S)-E_L(1S)$ is known theoretically to a higher precision than the values of the $1S$ and $2S$ Lamb shifts themselves (see also discussion in [28]).

Self-consistent values for the $1S$ Lamb shift in Table III have somewhat larger errors than the "experimental" results in the same Table, however, they do not depend on the experimental value of the $2S_{1/2} - 2P_{1/2}$ Lamb shift and on the value of the proton charge radius. The accuracy of the self-consistent numbers is mainly determined by the accuracy of the frequency measurements in [30,31,33]. Factor 4-5 reduction of the experimental errors would lead to a self-consistent determination of the $1S$ Lamb shift with the same accuracy as the accuracy of the "experimental" values cited in Table III. One may even invert the usual approach and extract values of the $2S_{1/2} - 2P_{1/2}$ Lamb shift from the respective self-consistent $1S$ values (see three last lines in Table II).

TABLE III

1S Lamb Shift

ΔE (kHz)	
8 172 860 (60)	Experimental value, Ref.[30].
8 172 815 (70)	Experimental value, Ref. [31].
8 172 844 (55)	Experimental value, Ref. [33].
8 172 915 (129)	Self-consistent value, Ref. [30].
8 172 763 (117)	Self-consistent value, Refs. [31,32].
8 172 858 (107)	Self-consistent value, Ref. [33].
8 172 729 (28) (32)	Theory, this work.

Recent theoretical development opens new ways to a more precise determination of the Rydberg constant, besides the one adopted now (see, e.g., [31,32]). First, one can use the self-consistent values of the $1S$ and $2S_{1/2} - 2P_{1/2}$ Lamb shifts to get the value of the Rydberg constant. Today such approach leads to a loss of accuracy (see Table IV , where the first error in the self-consistent values is determined by the accuracy of the self-consistent Lamb shift values and the second is determined by the accuracy of the frequency measurement), but greater accuracy may be achieved in future. Important advantage of such approach is that the value obtained in this way is independent of the direct experimental results on $2S_{1/2} - 2P_{1/2}$ Lamb shift and of the value of the proton charge radius. Second new approach is simply to reject the experimental data on the Lamb shifts and to use for the determination of the Rydberg constant directly the data on the frequencies of transitions between the levels with different main quantum numbers. Such approach becomes feasible now since the accuracy of the theoretical formulae for the frequences of the transitions is determined by the theoretical error of the expression for the $1S$ (or $2S$) Lamb shift which is about 28 kHz (and is even smaller for the $2S$ Lamb shift) and is thus smaller than the experimental error of the frequency determination. Respective values of the Rydberg constant derived from independent experimental data [31,32] are presented in the two last lines in Table IV, where the first error is determined by the accuracy of the theoretical expression, the second is defined by the experimental error of the frequency measurement, and the third one is determined by the experimental error in the determination of the proton charge radius. These values are pretty consistent, they are more accurate than the ones obtained by other methods and are the most precise contemporary values. Natural drawback of this approach is, of course, the dependence of the obtained value of the Rydberg constant on the proton charge radius.

TABLE IV

Rydberg Constant

R_∞ (cm^{-1})	
109 737. 315 684 1 (42)	Experimental value, Ref. [32].
109 737. 315 683 4 (24)	Experimental value, Ref. [31].
109 737. 315 686 8 (58) (20)	Self-consistent value, Refs. [32,30].
109 737. 315 681 1 (52) (14)	Self-consistent value, Refs. [31,32].
109 737. 315 679 7 (12)(20) (14)	Theory, this work, and Ref. [32].
109 737. 315 680 2 (05) (14) (06)	Theory, this work, and Ref. [31].

In conclusion we would like to emphasize that the high accuracy of the Lamb shift theory opens new perspectives in determination of the Rydberg constant and of the Lamb shift in the $1S$- and $2S$-states. Four directions of experimental investigations, namely, more precise measurement of the transitions between levels with different main quantum numbers, more precise measurement of the $1S$ and $2S$ Lamb shifts, and direct measurement of the proton charge radius seem especially promising. All these experiments are mutually complementary, since they lead to the values of the Rydberg constant of comparable accuracy based on the different kinds of experimental data. On the theoretical side calculation of the still unknown corrections to the energy levels with the goal of reduction of the theoretical error in the determination of the $1S$ Lamb shift to the level of 1 kHz (and, respectively, of the $2S$ Lamb shift to several tenth of kHz) seems to be both quite perspective and feasible.

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