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THE MULTISITE ANTIFERROMAGNETIC ISING SPIN MODEL AND THE UNIVERSALITY OF FEINGENBAUM EXPONENTS

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The Feingenbaum exponents α and δ for the tree-site antiferromagnetic Ising spin model on Husimi tree are calculated. It is shown that the numerical values of these exponents for this real statistical system coincide with the famous universal Feigenbaum exponents with high accuracy. The quantitative describtion of the transition from order to chaos is also abtained.

1. The concepts Scaling and Universality have played an essential role in the description of statistical systems [1].

Recently a multisite interaction system on Husimi tree approximation was investigated [2]. First, it was shown, that this approach yields good approximation for the ferromagnetic phase diagrams, which closely match the exact results obtained on a Kagome lattice [3]. Second, a multisite antiferromagnetic interaction was studied and interesting connections with the area of dynamical systems was made and the qualitative picture of full doubling bifurcation diagram including chaos, period-3 windows, etc., for the magnetization of the base site of this system was exhibited, whereas in antiferromagnetic Potts model only one period doubling occurred [4].

On the other hand, it is well known, that universality of Feigenbaum exponents directly applies to period doubling bifurcation sequence [5].

Note also, that in the anisotropic-next-nearest-neighbor Ising model on a Cayley tree in the infinite coordination limit the existence of chaotic phases associated with strange attractors have been obtained [6, 7].

The aim of our paper is numerical calculation of the Feigenbaum exponents for the three-site antiferromagnetic interaction (TSAI) Ising spin system on the Husimi tree with finite coordination number and to obtain the quantitative description of the transition from order to chaos for this statistical physical system.

2. The pure Husimi tree [8] shown in figure is characterized the γ-the number of the triangles,which go out of each site. The 0th-generation is a single central triangle.

The TSAI model in the magnetic field defined by the Hamiltonian

$$H = -J_3' \sum_{\Delta} \sigma_i \sigma_j \sigma_k - h' \sum_i \sigma_i, \qquad (1)$$

where σ_i takes values ± 1, the first sum goes over all triangular faces of the Husimi tree and the second over all sites. Besides we denote $J_3 = \beta J_3'$, $h = \beta h'$, $\beta = 1/kT$, where h – external magnetic field, T – temperature of the system and $J_3 < 0$ corresponds to the antiferromagnetic case.

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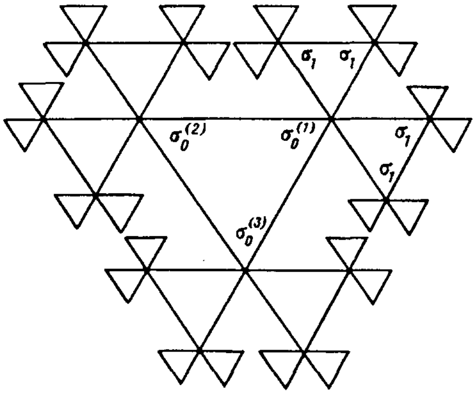

Husimi tree with $\gamma = 3$

The partition function will be written as

$$Z = \sum_{\{\sigma\}} \exp\left\{ J_3 \sum_{\triangle} \sigma_i \sigma_j \sigma_k + h \sum_i \sigma_i \right\}, \tag{2}$$

where the summation goes over all configurations of system.

The advantage of the Husimi tree introduced is that for the models formulated on it, exact recursion relation can be derived. When "cutting apart" the Husimi tree at the central triangle it separates into 3 identical branches and each of them contains $\gamma - 1$ branches. Then the partition function mey be written

$$Z = \sum_{\{\sigma_0\}} \exp\left\{ J_3 \sum_{\triangle} \sigma_0^{(1)} \sigma_0^{(2)} \sigma_0^{(3)} + h \sum_j \sigma_0^{(j)} \right\} [g_n(\sigma_0^{(1)})]^{\gamma-1} [g_n(\sigma_0^{(2)})]^{\gamma-1} [g_n(\sigma_0^{(3)})]^{\gamma-1}, \tag{3}$$

where $\sigma_0^{(j)}$ are spins of central triangle, n – number of shells and the equation for one of branches can be written:

$$g_n(\sigma_0) = \sum_{\{\sigma_i \neq \sigma_0\}} \exp\left\{ J_3 \sum_{\triangle} \sigma_0 \sigma_1 \sigma_1 + h \sum \sigma_1 + J_3 \sum_{\triangle} \sigma_i \sigma_j \sigma_k + h \sum_i \sigma_i \right\}. \tag{4}$$

One of branches, in its turn, can be cut on the site of 1th-generation, which is the nearest to the central site. Therefor, the expression for $g_n(\sigma_0)$ can be rewritten in the form:

$$g_n(\sigma_0) = \exp\left\{ J_3 \sum_{\triangle} \sigma_0 \sigma_1 \sigma_1 + h \sum \sigma_1 \right\} [g_{n-1}(\sigma_1^{(1)})]^{\gamma-1} [g_{n-1}(\sigma_1^{(2)})]^{\gamma-1}. \tag{5}$$

From eq.(5) one can easily obtain:

$$g_n(+) = e^{J_3+2h} g_{n-1}^{\gamma-1}(+) g_{n-1}^{\gamma-1}(+) + 2e^{-J_3} g_{n-1}^{\gamma-1}(+) g_{n-1}^{\gamma-1}(-) + e^{J_3-2h} g_{n-1}^{\gamma-1}(-) g_{n-1}^{\gamma-1}(-),$$

$$g_n(-) = e^{-J_3+2h} g_{n-1}^{\gamma-1}(+) g_{n-1}^{\gamma-1}(+) + 2e^{J_3} g_{n-1}^{\gamma-1}(+) g_{n-1}^{\gamma-1}(-) + e^{-J_3-2h} g_{n-1}^{\gamma-1}(-) g_{n-1}^{\gamma-1}(-).$$

Let the following variable be introduced:

$$x_n = \frac{g_n(+)}{g_n(-)}. \tag{6}$$

Then for x_n we can obtain the following recursion relation:

$$x_n = f(x_{n-1}), \qquad f(x) = \frac{z\mu^2 x^{2(\gamma-1)} + 2\mu x^{\gamma-1} + z}{\mu^2 x^{2(\gamma-1)} + 2z\mu x^{\gamma-1} + 1}, \tag{7}$$

where $z = e^{2J_s}$, $\mu = e^{2h}$ and $0 \le x_n \le 1$. The function $f(x)$ is unimodal: continuous, continuously differentiable and has one maximum x^* in $[0,1]$. Note that $f(x^*) = 1$ for any γ, h and T. The eq.(7) coincides with that obtained by Monroe [2], when pair interaction absents.

For magnetization of the central base site we obtain:

$$m = \langle \sigma_0 \rangle = \frac{e^h g_n^\gamma(+) - e^{-h} g_n^\gamma(-)}{e^h g_n^\gamma(+) + e^{-h} g_n^\gamma(-)} = \frac{e^h x_n^\gamma - 1}{e^h x_n^\gamma + 1}. \tag{8}$$

3. As it is mentioned above, the TSAI system is the nonlinear dynamical system and the qualitative picture of full doubling bifurcation diagrams, chaos etc., for the magnetization of the base site of it was existed [2].

The questions we want to address in this paper are, how calculate the exponents of Feigenbaum for TSAI system and if calculated values will coincide with the famous universal Feigenbaum exponents:

$$\alpha = 2.50290\ldots, \qquad \delta = 4.669201\ldots. \tag{9}$$

Feigenbaum observed for logistic map (see ref.[9]) two kinds of scaling: one that the length 2^n cycle first appears at a r_n value, which obeys:

$$r_n = r_\infty - \text{const}\,\delta^{-n}, \qquad n \gg 1, \tag{10}$$

where r_∞ the value of r from which the chaotic behavior ensues and the sequence essentially never repeats itself.

The other scaling was a special behavior which occurred near the x^* – value for which the map is extremal ($x^* = 1/2$ in the logistic map). If one started out at value for x^* then

$$-\alpha = \frac{d_n}{d_{n+1}}, \qquad n \gg 1, \tag{11}$$

where

$$d_n = f_{R_n}^{2^{n-1}}(x^*) - x^*. \tag{12}$$

In eq.(12) R_n are the values of r ($r_1 < R_1 < \ldots < R_n < r_n$) and

$$f_{R_n}^{2^n}(x^*) = x^*. \tag{13}$$

Note, that values of R_n and r_n have the same scale and $r_\infty = R_\infty$. Therefore

$$R_\infty = R_n - \text{const}\,\delta^{-n}. \tag{14}$$

Eqs.(11) and (14) defines two Feigenbaum exponents, which turns out to be "universal".

Now let us turn to our questions. One can see from eq.(7), that the role of above mentioned parameter r for TSAI system on Husimi tree for each fixed temperature plays external magnetic field h. The recursion function of eq.(7) has one maximum at $x^* = 1/\sqrt[\gamma-1]{\mu}$. Note, that this x^* depend on values of T and h, whereas in case of logistic map it is a constant. Further, we numerically solve the eq.(13) and find out the values of H_n (H_n is the analog of R_n). Using this values of H_n and eqs.(11) and (14) we calculate the Feigenbaum exponents for TSAI system. All our numerical calculations are done for $\gamma = 3$, $T = 0.3$, $J_3 = -1$, and are following:

period doubling	H_n	α	δ	magnetization m	x_n
$2^1 = 2$	0.18354515...			-0.6782977	0.5423560
				0.09151673	0.999999
$2^2 = 4$	0.28692571...	4.86428158...	3.50752342...		
$2^3 = 8$	0.31861247...	2.19505287...	4.32097441...	-0.8924331	0.3457500
				-0.1373881	0.8200590
				-0.9581313	0.2495890
				0.1182513	0.9733620
				-0.8506408	0.3886090
				-0.2286943	0.7699640
				-0.9640809	0.2369180
				0.1579719	0.9999
$2^4 = 16$	0.32607381...	2.76000232...	4.5870529...		
$2^5 = 32$	0.32770666...	2.42990139...	4.65118547...		
$2^6 = 64$	0.32805801...	2.5381186...	4.66503325		
$2^7 = 128$	0.32813334...	2.4897987...	4.66830065...		
$2^8 = 256$	0.32814947...	2.5099532...			
....	....	....	....	....	....
$2^\infty = \infty$	0.3281538...				

For const., presented in eq.(14), which is depend on family of reflection functions [10], for TSAI system we obtain the following value: const $= 0.99\ldots$, whereas for logistic map it is $0.12\ldots$.

Using the values of H_n and corresponding them $x_1, x_2, \ldots x_n$, we calculate the magnetization for base site (for each cycle of period doubling) of this system by eq.(8) as well. It means that each 2^n period doubling have n values of magnetization, which should be explained as an arising of a n-sublattice phase such that $x_1, x_2, \ldots x_n$ determine the states on each sublattice.

One can see from numerical results, that for real statistical physical system obtained values of α and δ coincide with famous Feigenbaum exponents (eq.(9)) with high accuracy and thereby confirm they universality once more.

The same Feingenbaum exponents for all temperatures below T^* are also calculated with high accuracy. $T^* \approx 0.55$ is the upper bound of chaotic temperature for TSAI system when $\gamma = 3$, $J_3 = -1$.

It is interesting to note, that if one lets $\gamma = 2$ in eq.(7) rather then $\gamma = 3$, the above mentioned situation changes dramatically.

Let us consider the following system of equations:

$$\begin{cases} f(x) - x = 0 \\ f'(x) = -1 \end{cases}. \tag{15}$$

The eq.(15) determined the point, where the first doubling bifurcation is begun.

For recursion function when $\gamma = 2$, the eq.(15) will have the form:

$$\begin{cases} \mu^2 x^3 + z\mu(2-\mu)x^2 + (1-2\mu)x - z = 0 \\ \mu^2 x^2 - 2z\mu^2 x - (1+2\mu) = 0 \end{cases}, \tag{16}$$

which for any T and h have only nonphysical solutions. Therefore, for TSAI system when $\gamma = 2$ the period doubling bifurcations picture absents. It means that for this statistical physical system there is not phase transition of second order when $\gamma = 2$.

4. In this paper we have investigated TSAI Ising spin model by approximating it with Husimi tree structures and calculate the Feigenbaum exponents α and δ. The numerical results show, that obtained valued for these exponents for real physical system coincide with the famous universal Feigenbaum exponents with high accuracy. The quantitative description of the transition from order to chaos is also obtained. Hence we see many of the very intensely studied and by now familiar properties of dynamical systems theory, which gives possibility to study the statistical physical systems in a new context and in a simple manner. In particular, in this paper with using the well known technique for dynamical systems, we analitically show, that for TSAI system there is not phase transition of second order when $\gamma = 2$.

We think, that obtained results are interesting and we plan to continue to investigate this line of approach for TSAI system and for several other systems.

On the other hand, the study of chaotic statistical physical system has opened new challenges for theories of stochastic processes, especially in the direction of stochasticity of vacuum in QCD [11]. In this direction the interesting results for Z(Q) gauge model on generalized Bethe lattice was obtained [12]. It gives bases to suppose that TSAI Ising spin model on Husimi tree approximation can be connected via duality [13] with n plaquette representation of the gauge theory.

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