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SOLITON MANAGEMENT IN THE NONLINEAR SCHRÖDINGER EQUATION MODEL WITH VARYING DISPERSION, NONLINEARITY AND GAIN

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The novel stable "soliton islands" in a "sea of solitary waves" of the nonlinear Schrödinger equation model with varying dispersion, nonlinearity and gain or absorption are discovered. Different soliton management regimes are predicted.

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Nonlinear Schrödinger equation model (NLSE) is one of the most important and "universal" nonlinear models of modern science. NLSE appears in many branches of physics and applied mathematics, including condensed matter and plasma physics, nonlinear optics and quantum electronics, fluid mechanics, theory of turbulence and phase transitions, biophysics, and star formation. The best-known solutions of the NLSE are those for solitary waves, or solitons. The theory of NLSE solitons was developed for the first time in 1971 by Zakharov and Shabad [1]. Zakharov and Shabat first applied the inverse scattering transform method to this equation and derived a more general form for its bright, dark and multisoliton solutions [1, 2]. Over the years, not only was experimental verification of the existence of NLSE solitons in many branches of modern science demonstrated, but also various properties of solitons derivable from the result of the inverse scattering transform theory were identified. Hasegawa and Tappert [3] were the first to show theoretically that an optical pulse in a dielectric fiber forms a solitary wave based on the fact that the wave envelope satisfies the NLSE. Optical solitons were discovered experimentally in 1980 by Mollenauer, Stolen and Gordon [4]. Today optical solitons are regarded as the natural data bits and as an important alternative for the next generation of ultra-high speed optical telecommunication systems [5]. Picosecond optical solitons theory, developed in the frame of the NLSE model, has produced an excellent agreement between theory and experiment [6, 7].

The problem of soliton management in the nonlinear systems described by the NLSE model with varying coefficients is a new and important one (see, for example, the review of optical solitons dispersion management principles and research as it currently stands in [8-10], and references therein). We should also note the fact that the first soliton dispersion management experiment in a fiber with hyperbolically decreasing group velocity dispersion was realized as earlier as in 1991 by Dianov's group in General Physics Institute [11].

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In this Letter we predict the existence of a new type of temporal and spatial solitary waves for the NLSE model with varying dispersion, nonlinearity and gain or absorption. We will then turn our attention to finding solutions for specified soliton management conditions. Different soliton management regimes are predicted.

Our starting point is the NLSE model with varying coefficients:

$$i\frac{\partial q^{\pm}}{\partial Z} \pm \frac{1}{2}D_2(Z)\frac{\partial^2 q^{\pm}}{\partial T^2} + N_2(Z) \mid q^{\pm} \mid^2 q = i\Gamma(Z)q^{\pm}. \quad (1)$$

NLSE (1) is written here in standard soliton units, as they are commonly known. There it is assumed that the perturbations to the dispersion parameter $D_2(Z)$, nonlinearity $N_2(Z)$ and to the amplification or absorption coefficient $\Gamma(Z)$ *are not limited* to the regime where they are smooth and small. Due to the well known spatial-temporal analogy [1] both temporal and spatial solitons are described by Eq.(1). In the case of temporal solitons T is the nondimentional time in the retarded frame associated with the group velocity of wave packets at a particular optical carrier frequency. In the case of two-dimensional spatial solitons $T = X$ represents a transverse coordinate.

Theorem 1. Consider the NLSE model (1) with varying dispersion, nonlinearity and gain or absorption. Suppose that the Wronskian $W\{N_2, D_2\}$ of the functions $N_2(Z)$ and $D_2(Z)$ is nonvanishing, the two functions $N_2(Z)$ and $D_2(Z)$ are thus linearly independent. There are then an infinite number of a solitary wave solutions for the Eq. (1) written in the following form:

$$q^{\pm}(Z,T) = \sqrt{\frac{D_2(Z)}{N_2(Z)}}P(Z)Q^{\pm}\left[P(Z)T\right]\exp\left[\pm i\frac{P(Z)}{2}T^2 + i\int_0^Z K^{\pm}(Z')dZ'\right], \quad (2)$$

where the real function $Q^{+}(S)$ describes canonical functional form of bright (sign=+1 , $Q^{+}(S) = \eta\mathrm{sech}(\eta P(Z)T)$) or dark (sign=-1 , $Q^{-}(S) = \eta\tanh(\eta P(Z)T)$) NLSE solitons [1-3], and the real functions $D_2(Z)$, $N_2(Z)$, $\Gamma(Z)$, and $P(Z)$ satisfy the equation system:

$$\frac{\partial P(Z)}{\partial Z} + P^2(Z)D_2(Z) = 0; \frac{W\{N_2(Z), D_2(Z)\}}{D_2(Z)N_2(Z)} - D_2(Z)P(Z) = 2\Gamma(Z). \quad (3)$$

Theorem 2. Consider the NLSE model (1) with varying dispersion, nonlinearity and gain or absorption. Suppose that the Wronskian $W\{N_2, D_2\}$ is vanishing, the two functions $N_2(Z)$ and $D_2(Z)$ are thus linearly dependent. There are then an infinite number conserving the pulse area solitary wave solutions for the Eq.(1):

$$q^{\pm}(Z,T) = CP(Z)Q^{\pm}\left[P(Z)T\right]\exp\left[\pm i\frac{P(Z)}{2}T^2 + i\int_0^Z K^{\pm}(Z')dZ'\right], \quad (4)$$

where the real functions $Q^{\pm}(S)$ describe a canonical form of bright ($Q^{+}(S)$) or dark ($Q^{-}(S)$) NLSE solitons, and the real functions $P(Z)$, $D_2(Z)$, $N_2(Z)$ and $\Gamma(Z)$ satisfy the equation system:

$$2\Gamma(Z) = \frac{1}{P}\frac{\partial P(Z)}{\partial Z}; \quad C^2N_2(Z) = D_2(Z) = -\frac{1}{P^2(Z)}\frac{\partial P(Z)}{\partial Z}. \quad (5)$$

The explicit solutions for the travelling solitary waves can easily be constructed by applying the Galilei transformation and by using the equation for the "soliton center" $T_{sol}(Z)$ given by

$$\partial T_{sol}(Z)/\partial Z = -VD_2(Z), \tag{6}$$

where V is a soliton group velocity (in the case of spatial soliton $V = \tan\theta$, and θ is the angle of propagation in the $X - Z$ plane).

By applying Theorems 1 and 2 we develope a systematic analytical approach to find the fundamental set of the different NLSE solitons management regimes.

Case 1. Soliton dispersion management. In this case the dispersion management function $D_2(Z)$ is assumed to be given: $D_2(Z) = \Phi(Z)$ (we call it control function here). The function $\Phi(Z)$ is required only to be a once-differentiable and once integrable, but otherwise arbitrary function, there are no restrictions. There are then an infinite number of solutions for the Eq.(1) of the form of bright and dark dispersion managed solitons represented by the Eq.(2), where the main functions $P(Z)$ and $\Gamma(Z)$ are given by

$$P(Z) = -\frac{1}{[C - \int \Phi(Z)dZ]} \quad \Gamma(Z) = \frac{1}{2}\frac{\partial}{\partial Z}\ln\left(\left|\frac{P(Z)\Phi(Z)}{N_2(Z)}\right|\right). \tag{7}$$

In the limit of $N(Z) = \text{const}$ Eq.(7) reduces to:

$$\Gamma(Z) = \frac{1}{2}\frac{\Phi(Z)}{[C - \int \Phi(Z)dZ]} + \frac{1}{2}\frac{1}{\Phi(Z)}\frac{\partial\Phi(Z)}{\partial Z}, \tag{8}$$

where C is the constant of integration.

Case 2. Soliton energy control. In this case the soliton energy control function $E(Z) = 2D_2(Z)P(Z)/N_2(Z)$ is assumed to be given. The function $E(Z)$ is required only to be a once-differentiable and once integrable, but otherwise arbitrary function, there are no restrictions. There are then an infinite number of solutions for the Eq.(1) of the form of bright and dark solitons represented by the Eq.(2), where the main functions $D_2(Z)$, $P(Z)$, and $\Gamma(Z)$ are given by

$$D_2(Z) = \frac{E(Z)N_2(Z)}{2P(Z)} \quad 2\Gamma(Z) = \frac{\partial}{\partial Z}\ln(E(Z)/2), \tag{9}$$

$$P(Z) = \pm\exp\left[-\frac{1}{2}\int E(Z)N_2(Z)dZ + C\right]. \tag{10}$$

Case 3. Soliton intensity management. In this case the soliton pulse intensity (peak power) is assumed to be controlled by the function $\Theta(Z) = D_2(Z)P^2(Z)/N_2(Z)$, where the control function $\Theta(Z)$ is required only to be a once-differentiable and once integrable. There are then an infinite number of solutions for the Eq.(1) of the form of bright and dark solitons represented by the Eq.(2), where the main functions $D_2(Z)$, $P(Z)$, and $\Gamma(Z)$ are given by quadratures:

$$D_2(Z) = \frac{\Theta(Z)}{[C - \int \Theta(Z)dZ]^2}; \quad P(Z) = -\int \Theta(Z)dZ + C,$$

$$2\Gamma(Z) = \frac{\Theta(Z)}{[C - \int \Theta(Z)dZ]} + \frac{1}{\Theta(Z)}\frac{\partial\Theta(Z)}{\partial Z} \tag{11}$$

and the nonlinearity is assumed to be a constant ($N_2(Z) \equiv 1$).

Case 4. Soliton pulse width management and the problem of optimal soliton compression. In this case the soliton pulse width control function is assumed to be given: $\Upsilon(Z) = P^{-1}(Z)$. The real function $\Upsilon(Z)$ is required only to be a twice-differentiable, but otherwise arbitrary function, there are no restrictions. There are then an infinite number of solutions for the Eq.(1) of the form of bright and dark solitons represented by the Eq.(2), where the main coefficients of the NLSE model $D_2(Z)$ and $\Gamma(Z)$ are given by

$$D_2 = \frac{\partial \Upsilon}{\partial Z}; 2\Gamma = -\frac{1}{\Upsilon}\frac{\partial \Upsilon}{\partial Z} + \left(\frac{\partial \Upsilon}{\partial Z}\right)^{-1}\frac{\partial^2 \Upsilon}{\partial Z^2}. \tag{12}$$

Case 5. Soliton amplification management and the problem of optimal soliton amplification. In this case the gain (or loss) function $\Gamma(Z)$ is assumed to be given: $\Gamma(Z) = \Lambda(Z)$. The gain control function $\Lambda(Z)$ is required only to be once integrable. There are then an infinite number of solutions for the Eq.(1) of the form of bright and dark solitons represented by the Eq.(2), where the main functions $D_2(Z)$ and $P(Z)$ are given by quadratures:

$$|P(Z)|\,|D_2(Z)| = \exp\left[\int 2\Lambda(Z)dZ + C_1\right], \tag{13}$$

$$|D_2| = \exp\left\{\int \left[2\Lambda(Z) \pm |P(Z)|\,|D_2(Z)|\right] dZ + C_2\right\}, \tag{14}$$

where the integration constants $C_{1,2}$ are determined by initial conditions.

Case 6. Combined nonlinear and dispersion soliton management regimes. In this case the Wronskian $W\{N_2, D_2\}$ is assuming to be vanishing, that means the nonlinearity and dispersion are linearly dependent functions. The main feature of soliton solutions given by Theorem 2 consists in the fact that the soliton pulse area is conserved during propagation. Suppose that the dispersion management function $D_2(Z)$ is determined by the known control function $D_2(Z) = \Xi(Z)$, where the function $\Xi(Z)$ is required only to be a once integrable. There are then an infinite number of solutions for the Eq.(1) of the form of bright and dark conserving pulse area dispersion managed solitons represented by the Eq.(4), where the main functions $D_2(Z)$, $P(Z)$, $N_2(Z)$ and $\Gamma(Z)$ are given by quadratures:

$$P(Z) = -1/\left[C - \int \Xi(Z)dZ\right] \quad N_2(Z) = D_2(Z)/C^2, \tag{15}$$

$$2\Gamma(Z) = \Xi(Z)/\left[C - \int \Xi(Z)dZ\right]. \tag{16}$$

The interested reader can take different control functions $\Phi(Z)$ (Eqs.(7), (8)); $E(Z)$ (Eqs.(9), (10); $\Theta(Z)$ (Eqs.(11)); $\Upsilon(Z)$ (Eqs.(12); $\Lambda(Z)$ (Eqs.(13), (14)) and $\Xi(Z)$ (Eqs.(15), (16) to find the novel "soliton islands" in a "sea of solitary waves" for the NLSE model (1) by using algorithms developed in this work. Soliton management scenario is being determined by the indefinite integrals in Eqs.(7)–(16) which are elementary for practically any of the best-known elementary functions (considered here as a probe control or management functions $\Phi(Z)$.....$\Xi(Z)$): rational, algebraic, the exponential and hyperbolic, trigonometric and logarithms, and their combinations. We will present the most interesting (from the application point of view) examples in a separate publication.

The main soliton features of analytical solutions predicted have been investigated by using direct computer simulations with the accuracy as high as 10^{-9}. In the future publications we will show that managed solitary waves predicted not only interact elastically, but they can form the bound states, and these bound states split under weak perturbations.

Recently Zakharov and Manakov [12] showed that in the strong dispersion managed nonlinear system the leading nonlinear effect is the formation of a collective average dispersion which is a result of the interaction of all soliton pulses propagating along the optical fiber communication line, and, in the leading order, the system is described by an integrable Hamiltonian system with the plethora of soliton solutions. It was shown that due to formation of an additional collective dispersion each pulse in the line generates long tails that influence the shapes of the other pulses, the pulses feel each other when separated by an arbitrary long distance [12].

The methodology developed in this Letter (Theorems 1 and 2) provides for a systematic way to discover and investigate another class of the managed solitons with canonical bright and dark soliton pulses profiles. The surprising aspect is that an analytical solutions are obtained here in quadratures. Their pure soliton-like features are confirmed by the accurate direct computer simulations. We should also note that solitary waves for the NLSE model (1) must be of rather general character than canonical solitons for the standard NLSE model with constant coefficients, because the generalized model (1) takes into account arbitrary variations of group velocity dispersion $D_2(Z)$, nonlinearity $N_2(Z)$ and gain (or absorption) $\Gamma(Z)$. The results obtained in this Letter are of general physics interest and should be readily experimentally verified.

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1. V.E.Zakharov and A.B.Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov.Phys.JETP, **34**, 62 (1972)].
2. V.E.Zakharov and A.B.Shabat, Zh. Eksp. Teor. Fiz. **64**, 1627 (1973) [Sov. Phys JETP **37**, 823 (1973)].
3. A.Hasegawa and F.Tappert, Appl. Phys. Lett. **23**, 142, (1973); **23**, 171 (1973).
4. L.F.Mollenauer, R.G.Stolen, and J.P.Gordon, Phys. Rev. Lett. **45**, 1095 (1980).
5. A.Hasegawa and Y.Kodama, *Solitons in Optical Communications*, Clarendon, Oxford, 1995.
6. J.R.Taylor, Ed., *Optical solitons – theory and experiment*, Cambridge Univ. Press, 1992.
7. G.P.Agrawal, *Nonlinear Fiber Optics*, Academic, New York, 1995.
8. A.Hasegawa, Physica **D123**, 267 (1998).
9. T.I.Lakoba and D.J.Kaup, Phys. Rev. **E58**, 6728 (1998).
10. S.K.Turitsyn and E.G.Shapiro, J. Opt. Soc. Am. **B16**, 1321 (1999).
11. V.A.Bogatyrev, M.M.Bubnov, E.M.Dianov et al., J. Lightwave Technol **9**, 561 (1991).
12. V.E.Zakharov and S.V.Manakov, Pis'ma Zh. Eksp. Teor. Fiz. **70**, 573 (1999) [JETP Lett. **70**, 578 (1999)].