

ON THE BOUNDARY OF THE DISPERSION-MANAGED SOLITON EXISTENCE

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A breathing soliton-like structure in dispersion-managed optical fiber system is studied. It is proven that for negative average dispersion the breathing soliton is forbidden provided that a modulus of average dispersion exceed a threshold which depends on the soliton amplitude.

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Propagation of optical pulse in nonlinear media with varying dispersion is both fundamental [1] and important applied problem [2–8] because a dispersion-managed (DM) system, which is a system with periodic dispersion variation along an optical fiber, is one of the most prospective candidate for ultrafast high-bit-rate optical communication lines. Lossless propagation of optical pulse in DM fiber is described by a nonlinear Schrödinger equation (NLS) with periodically varying dispersion $d(z)$:

$$iu_z + d(z)u_{tt} + |u|^2u = 0, \quad (1)$$

where u is the envelope of the optical pulse, z is the propagation distance and all quantities are made dimensionless. Consider a two-step periodic dispersion map: $d(z) = d_0 + \tilde{d}(z)$, where $\tilde{d}(z) = d_1$ for $0 < z + nL < L_1$ and $\tilde{d}(z) = d_2$ for $L_1 < z + nL < L_1 + L_2$, d_0 is the path-averaged dispersion, d_1, d_2 are the amplitudes of dispersion variation subject to a condition $d_1L_1 + d_2L_2 = 0$, $L \equiv L_1 + L_2$ is a dispersion compensation period and n is an arbitrary integer number. Eq. (1) also describes pulse propagation in a fiber with losses compensated for by periodically placed amplifiers if the distance between amplifiers is much less than L .

In a linear regime, in which the nonlinear term in Eq. (1) is negligible, the periodical variation of dispersion is a way to overcome pulse broadening due to the chromatic dispersion provided that the residual dispersion d_0 is small enough. However in real optical fiber the nonlinear term in (1) is important because the optical pulse amplitude should be big enough to get high signal/noise ratio. One of the fascinating feature of DM system is the numerical observation of a space-breathing soliton-like structure, which is called DM soliton, for both positive and negative residual dispersion d_0 [9]. This observation is in sharp contrast with the system described by NLS with the constant dispersion where stable soliton propagation is possible only for the positive dispersion [10] because the nonlinearity can continuously compensate the positive dispersion only. In DM soliton the balance between the nonlinearity and dispersion is achieved on average over the dispersion period L what lift a restriction of the positive dispersion sign. Nevertheless it

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was never proven that DM soliton really exists because there is a possibility that this is rather a long-lived quasi-stable breathing pulse which decays on a long distance z . It is shown here that for negative d_0 DM soliton can exist only if $|d_0|$ is small enough to allow nonlinear compensation of pulse broadening due to the dispersion over distance L .

Eq. (1) can be written in the Hamiltonian form $iu_z = \delta H / \delta u^*$, where the Hamiltonian

$$H = \int \left[d(z)|u_t|^2 - \frac{|u|^4}{2} \right] dt, \quad (2)$$

is an integral of motion on each interval of a constant dispersion $d(z) = \text{const}$. Eq. (1) is reduced to usual NLS on such intervals. At points $z = nL$ and $z = nL + L_1$, where n is an arbitrary integer number, the Hamiltonian experiences jumps due to jumps of the dispersion although the value of u is a continuous function of z in these points. In contrast to the Hamiltonian the time-averaged optical power $N = \int |u|^2 dt$ or number of particles in the quantum mechanical interpretation of NLS (in this interpretation the coordinate z means some "time" and actual time t has a meaning of "coordinate") is an integral of motion for all z . Consider z -dependence of the quantity $A = \int t^2 |u|^2 dt$. A/N is the average width of a time-distribution of u or simply $\langle t^2 \rangle$ in a quantum mechanical interpretation of NLS.

Using (1) and integrating by parts one gets for the first z derivative

$$A_z = d(z) \int 2it(uu_t^* - u^*u_t) dt. \quad (3)$$

In a similar way after a second differentiation by z one gets

$$A_{zz} = 4dH + 4d^2X + \frac{d_z}{d}A_z, \quad (4)$$

where $X \equiv \int |u_t|^2 dt$. It follows from Eq. (3), which is often called virial theorem (see e.g. Refs.[11, 12]), that A_z experiences finite jumps corresponding to jumps of a step-wise function $d(z)$:

$$\begin{aligned} A_z \Big|_{z=L_1+0} &= \frac{d_0 + d_2}{d_0 + d_1} A_z \Big|_{z=L_1-0}, \\ A_z \Big|_{z=L+0} &= \frac{d_0 + d_1}{d_0 + d_2} A_z \Big|_{z=L-0}. \end{aligned} \quad (5)$$

Set $X(z) = X_0 + \delta X(z)$, $X(0) \equiv X_0$ then one can integrate Eq. (4) over intervals $(0, L_1)$, (L_1, L) :

$$\begin{aligned} A_z \Big|_{z=L_1-0} &= A_z \Big|_{z=0+0} + 4 \int_0^{L_1} [(d_0 + d_1)H_1 + (d_0 + d_1)^2 X] dz, \\ A_z \Big|_{z=L-0} &= A_z \Big|_{z=L_1+0} + 4 \int_{L_1}^L [(d_0 + d_2)H_2 + (d_0 + d_2)^2 X] dz, \end{aligned} \quad (6)$$

where

$$H_1 = (d_0 + d_1)X_0 - Y_0, \quad H_2 = (d_0 + d_2)X_0 - Y_0 - (d_1 - d_2)\delta X \Big|_{z=L_1} \quad (7)$$

are the Hamiltonian values on intervals $(0, L_1)$, (L_1, L) respectively,

$$Y(z) \equiv \int \frac{|u|^4}{2} dt, \quad Y_0 \equiv Y(0).$$

Here the conservation of H_1 on interval $(0, L_1)$ is used in derivation of expression for H_2 .

The DM soliton solution of Eq. (1) (see Ref. [13]) is given by $u = \tilde{u}(z, t) \exp(ikz)$, where k is an arbitrary real constant and $\tilde{u}(z + L, t) = \tilde{u}(z, t)$ is a periodic function of z , $\tilde{u}(z, t)|_{|t| \rightarrow \infty} \rightarrow 0$. Thus for DM soliton $A_z \Big|_{z=L+0} = A_z \Big|_{z=0+0}$. This condition can be cast via Eq. (5) - (7) into the form:

$$L(d_1 + d_0) [2d_0 X_0 - Y_0 + (d_1 - d_2) \frac{L_2}{L} \delta X \Big|_{z=L_1}] + \int_0^{L_1} (d_0 + d_1)^2 \delta X dz + \int_{L_1}^L (d_0 + d_2)^2 \delta X dz = 0. \quad (8)$$

Next step is to consider $\delta X(z)$ dependence. Using (1) and integrating by parts one can get

$$X_z = 4 \int \phi_t R_t R^3 dt, \quad (9)$$

where $u \equiv Re^{i\phi}$, ϕ and R are real, $R \geq 0$. Consider an upper bound of X_z which is given by a chain of inequalities:

$$4 \int \phi_t R_t R^3 dt \leq 4 \max_t (R^2) \int |\phi_t R_t R| dt \leq 4 X^{3/2} N^{1/2}, \quad (10)$$

where the following inequalities are used:

$$2\phi_t R_t R \leq (\phi_t R)^2 + R_t^2, \quad \max_t (R^2) \leq \int_{-\infty}^t |(R^2)_t| dt' \leq \int | (R^2)_t | dt \leq 2 \int R |R_t| dt \leq 2 N^{1/2} X^{1/2} \quad (11)$$

(in last expression the Cauchy - Schwarz inequality is also used). Eq. (9) and (10) can be integrated by z and give (it is assumed below that $2X_0^{1/2} N^{1/2} \max(L_1, L_2) < 1$):

$$X \leq X_0 / (1 - 2X_0^{1/2} N^{1/2} z)^2. \quad (12)$$

In a similar way using inequality $X_z \geq -4 \int |\phi_t R_t| R^3 dt$ following from (9) one can get the lower bond of $X(z)$:

$$X \geq X_0 / (1 + 2X_0^{1/2} N^{1/2} z)^2. \quad (13)$$

For DM soliton $X(L) = X_0$ and thus it is more convenient to use for $L_1 < z < L$ similar inequalities:

$$\frac{X_0}{(1 + 2X_0^{1/2} N^{1/2} (L - z))^2} \leq X \leq \frac{X_0}{(1 - 2X_0^{1/2} N^{1/2} (L - z))^2}. \quad (14)$$

Eqs. (8) - (14) result in inequality:

$$|2d_0 X_0 - Y_0| \leq \frac{|d_1 - d_2| L_2 X_0}{L} \left[\frac{1}{(1 - 2X_0^{1/2} N^{1/2} L_1)^2} - 1 \right] + \frac{2X_0^{3/2} N^{1/2}}{|d_0 + d_1| L} \left[\frac{(d_0 + d_1)^2 L_1^2}{1 - 2X_0^{1/2} N^{1/2} L_1} + \frac{(d_0 + d_2)^2 L_2^2}{1 - 2X_0^{1/2} N^{1/2} L_2} \right]. \quad (15)$$

Eq. (15) is the main result of the present paper. Eq. (15) is a consequence of initial assumption that DM soliton exist for given parameters L_1, L_2, d_0, d_1, d_2 and integral values

X_0, Y_0, N which depend on $u|_{z=0}$ only. Thus DM soliton can exist only if this inequality is fulfilled.

Note that if one assumes uniqueness of DM soliton solution for given k and soliton width then, as shown in Ref. [13], $|u|_{z=0} = |u|_{z=L_1}$. In such a case the term $\delta X|_{z=L_1}$ in Eq. (8) vanishes and instead of (15) one can get a stricter inequality. Here however that possibility is disregarded for the sake of generality.

To clarify physical consequences of Eq. (15) consider the optical pulse with a typical amplitude p and a typical time-width t_0 . Then $N \sim |p|^2 t_0$, $X_0 \sim |p|^2 / t_0$ and thus $X_0^{1/2} N^{1/2} L \sim L / Z_{nl}$, where $Z_{nl} = 1/|p|^2$ is a characteristic nonlinear length. In a typical experimental condition a nonlinearity is small: $L / Z_{nl} \ll 1$ and denominators in (15) can be series expanded thus giving

$$|2d_0 X_0 - Y_0| \leq \frac{2X_0^{3/2} N^{1/2}}{L} \left[2|d_1 - d_2| L_1 L_2 + |d_0 + d_1| L_1^2 + \frac{(d_0 + d_2)^2 L_2^2}{|d_0 + d_1|} \right] + O\left(\frac{d_1 L^3}{t_0 Z_{nl}^3}\right). \quad (16)$$

Provided that d_0 is negative both terms in left-hand side of (16) have the same sign and thus right-hand side should be greater or equal to $2|d_0| X_0 + Y_0$. Assuming $d_1 \gg |d_0|$ one can get from (16) the following estimate ($Y_0 \sim t_0 / Z_{nl}^2$):

$$\frac{2|d_0|}{t_0 Z_{nl}} + \frac{t_0}{Z_{nl}^2} \lesssim \frac{4L_1 d_1}{Z_{nl}^2 t_0} \left(1 + \frac{L_1}{L}\right). \quad (17)$$

Consider a strong dispersion management limit $Z_{disp} / L \ll 1$, where $Z_{disp} \equiv 2d_1 L_1 / t_0^2$ is a typical dispersion length. This limit implies that an optical pulse experiences strong oscillation on each period L due to dispersion. Then (17) reduces to

$$-\frac{d_0}{d_1} \lesssim \frac{6L_1}{Z_{nl}} \left(1 + \frac{L_1}{L}\right), \quad (18)$$

i.e. a nonlinearity (amplitude of the optical pulse) should be strong enough to allow DM soliton solution existence for a given negative d_0 .

Eq. (15) gives a necessary condition for DM soliton existence but not sufficient. In other words violation of the inequality (15) means that DM soliton is forbidden. Of course it would be interesting to find to what extent this necessary existence condition is close to sufficient one. In general this could be done only if one found DM soliton analytically.

Here one can only mention that there is a qualitative correspondence between threshold of DM soliton existence following from the analytical condition (15) and from a numerical investigation of DM soliton. Namely the maximal value of $|d_0|$ ($d_0 < 0$) for which DM soliton exist grows with increase of the dispersion map strength L / Z_{disp} according to both numerics (see e.g. [14, 15]) and the analytical condition (16). It also follows from (18) that for asymmetric dispersion map $L_1 \neq L_2$ maximal possible value of $|d_0|$ grows as L_1 increase (for fixed L, Z_{nl}, d_1) in correspondence with Fig. 3 of Ref. [15].

Eq. (15) has also a clear physical meaning in another limit $d_0 / d_1 \gg L / Z_{nl}, Z_{disp} \gg L$ and $Z_{nl} \gg L$ in which (15) reduces to:

$$(2d_0 X_0 - Y_0) / Y_0 = O(L / Z_{disp}) \ll 1. \quad (19)$$

Equality $2d_0 X_0 = Y_0$ exactly corresponds to one-soliton solution of NLS with dispersion d_0 (see Ref. [10]) where the dispersion d_0 and the nonlinearity continuously balance each other. Thus in the limit $Z_{disp} \gg L$, which is called a weak dispersion limit, we

recover usual NLS describing a path-averaged (over space period L) DM soliton dynamics provided d_0 is large enough. A weak dispersion management limit was studied earlier [1, 16–18]. Note that an additional condition $d_0/d_1 \gg L/Z_{nl}$ allows the amplitude d_1 of the dispersion variation still to be much higher than d_0 because one assumes $L \ll Z_{nl}$.

In conclusion the necessary analytical condition (15) of DM soliton existence is established. From a physical point of view this condition means that DM soliton solution can exist only if the nonlinearity is strong enough to compensate the pulse broadening due to the negative value of the average dispersion d_0 . Note that estimates in Eqs. (16)–(19) are only given here for a physical interpretation of the analytical condition (15). So far DM soliton solution was obtained numerically [3, 4, 14], by variational [5] and other perturbative approaches [19–21]. These results are in agreement with the condition (15). But analytical proof of DM soliton existence in the parameter region satisfying the condition (15), i.e. the sufficient existence condition, is still an open question.

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