Pis'ma v ZhETF, vol.72, iss.6, pp.493 - 498

REMARKS ABOUT THE EFFECTIVE CONDUCTIVITY OF SOME THREE-COLOR TESSELATIONS IN THE PLANE

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Submitted 15 August 2000

The asymptotic formula for the effective conductivity of the isotropic three-color (three-conductivity) rhombic tesselation in the plane is obtained for the case when one conductivity is much smaller than two others. The tentative formula for this rhombic tesselation is suggested and discussed.

PACS: 03.50.De, 41.20.-q, 02.30.Nw, 72.80.Tm

In spite of the fact that the history of studying the effective conductivity of composite materials is long enough [1], the number of exact mathematical results in this field is rather restricted. The most important from these results is the duality relation for the effective conductivity of the plane covered by the pieces of medium with different conductivities. This relation was obtained in the seminal paper by Keller [2] and then rederived by Dykhne [3] for the random equal-weighted distribution. Mendelson [4] has analyzed the applicability of this relation for the three-color non equal-weighted structures. For the general isotropic structure of plane tesselation or for an anisotropic structure, possessing the main axes, this relation can be written down as follows:

$$\sigma_{xx}(\sigma_1,\sigma_2,\ldots)\sigma_{yy}(\tilde{\sigma}^2/\sigma_1,\tilde{\sigma}^2/\sigma_2,\ldots)=\tilde{\sigma}^2, \tag{1}$$

where $\tilde{\sigma}$ is the arbitrary number of the corresponding dimensionality. Let us stress that the formula (1) is true also for non equal-weighted distributions of conductivities. For the two-color isotropic equal-weighted structures from this general duality relation (1) immediately follows well-known formula $\sigma=\sqrt{\sigma_1\sigma_2}$. For the anologous problem with three colors one can see that when $\sigma_3=\sqrt{\sigma_1\sigma_2}$ we immediately get that effective conductivity is equal to σ_3 [3]. Unfortunately, it is of little use for the construction of general formules for three-color tesselations.

Notice also, that the duality relation for three-color isotropic tesselations contains information about first and second partial derivatives of the effective condictivity in respect to the partial conductivities. Making calculations in the vicinity of the point when all the partial conductivities are equal (for the convenience, we choose them equal to 1), one can find that:

$$\frac{\partial\sigma}{\partial\sigma_i}=1/3, \tag{2}$$

$$\partial^2\sigma/\partial\sigma_i\partial\sigma_i=-2/9, \tag{3}$$

$$\partial^2\sigma/\partial\sigma_i\partial\sigma_k=1/9,\ i\neq k. \tag{4}$$

It is not surprising because in first two orders of the perturbation theory the structure is not essential and manifest itself beginning with third order.

In this letter, we would like to investigate some properties of the regular three-color (i.e. three-conductivity) structures on the plane. For the convenience we reproduce here the table of the so-called Dirichlet tesselations in the plane [5] (see Fig.1). From these tesselations one can get the most interesting equal-weighted three-color structures on the plane.

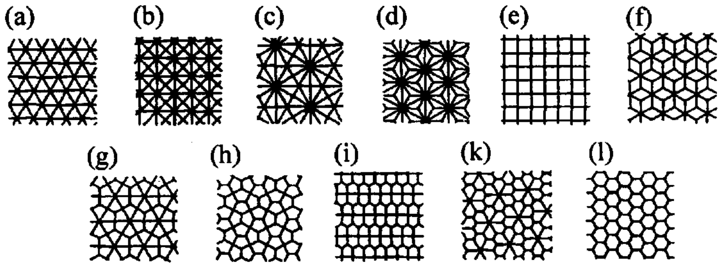

Fig.1. The 11 topological types of isohedral face-to face tesselations in the plane E^2. Notice that honeycomb structure l) is a mother one and all others can be obtained from it by contraction of edges or polygons

Beside the exact formulas and relations described above an important value have asymptotic relations which can be obtained for the cases when the conductivity of one component is much smaller when that of others. Important relations of this kind were obtained by Keller [6]. The key element of this consideration is the calculation of effective conductivity for the corner, which consists from four sectors. The medium with high conductivity σ_a occupies the sector $-\alpha/2 < \theta < \alpha/2$ and the opposite sector while the two other sectors contain the medium of a very small conductivity σ_b. For such a corner one has the following formula for the effective conductivity [6]:

$$\sigma(\alpha) \sim (\alpha\sigma_a\sigma_b/(\pi-\alpha))^{1/2}, \text{ for } \sigma_a/\sigma_b \gg 1. \tag{5}$$

This formula was used in the paper [6] for the investigation of the effective conductivity of the checkerboard covered by parallelograms and also for the generalization of the asymptotic formulas for two-color three-dimensional parallelepipedal structures studied before in [7].

Here, following the scheme elaborated in Ref. [6] we shall consider the corner with six alternated sectors (each of which is equal to $\pi/3$), three of which have a very high conductivity while other have a very small one. Looking at Fig.2, it is easy to see that this structure is basic one for the three-color tesselation in the plane by rhombs (see also Fig.1f).

Thus, we shall consider the circle of the radius 1, where the regions of a high conductivity σ_1 occupy the sectors $-\pi/6 < \theta < \pi/6$, $\pi/2 < \theta < 5\pi/6$ and $-5\pi/6 < \theta < -\pi/2$ (see Fig.3). Orienting the electric field along the axis $\theta = 0$ from the left to the right, we see that the current flows from the last two sectors to the first one. It is convenient to choose the boundary conditions in the following way: the potential φ on the circumference ($\rho = 1$) is equal to $+1$ at $-\pi/6 < \theta < \pi/6$ and is equal to -1 at $\pi/2 < \theta < 5\pi/6$ and $-5\pi/6 < \theta - < \pi/2$. Then an effective conductivity of the corner could be written down

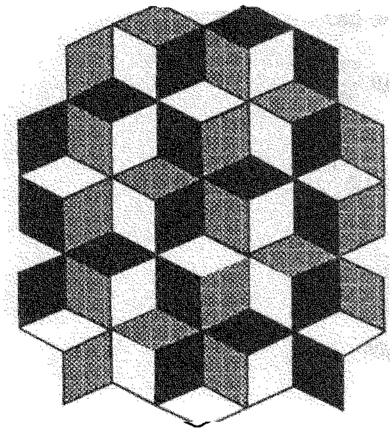

Fig.2. Three-color rhombic tesselation of the plane

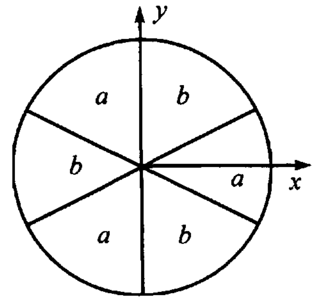

Fig.3. Basic corner for the rhombic tesselation

as

$$2\sigma = \int_{-\pi/6}^{\pi/6} \sigma_a \frac{\partial \varphi}{\partial \rho}(\rho, \theta)|_{\rho=1} d\theta = \sigma_a \int_{-\pi/6}^{\pi/6} \varphi_\rho(1, \theta) d\theta \tag{6}$$

or

$$\sigma = \sigma_a \int_0^{\pi/6} \varphi_\rho(1, \theta) d\theta. \tag{7}$$

We shall look for a solution of the Laplace equation for φ satisfying the following boundary conditions:

$$\varphi(1, \theta) = 1, \quad 0 < \theta < \pi/6, \tag{8}$$

$$\varphi_\rho(1, \theta) = 0, \quad \pi/6 < \theta < \pi/2, \tag{9}$$

$$\varphi_\theta(\rho, 0) = 0, \tag{10}$$

$$\varphi\left(\rho, \frac{\pi}{6}-\right) = \varphi\left(\rho, \frac{\pi}{6}+\right), \tag{11}$$

$$\varphi\left(\rho, \frac{\pi}{3}\right) = 0, \tag{12}$$

$$\sigma_a \varphi_\theta\left(\rho, \frac{\pi}{6}-\right) = \sigma_b \varphi_\theta\left(\rho, \frac{\pi}{6}+\right). \tag{13}$$

Here the Eq. (9) reflects the quasi-insulator nature of the second medium, Eq. (10) - the symmetry of the current in respect to the axis $\theta = 0$, Eq. (11) - the continuity of the potential on the boundary between two mediums while Eq. (13) describes the continuity of the current through this boundary. The Eq. (12) reflects the fact that due to the symmetry of the problem the line of vanishing potential is $\theta = \pi/3$. (Notice, that in the original problem with four-sector corner such a line was located at $\theta = \pi/2$ [6].

We shall look for the solution of the Laplace equation in the following form:

$$\varphi = A_a \rho^\nu \cos \nu\theta, \quad 0 < \theta < \frac{\pi}{6}, \tag{14}$$

and

$$\varphi = A_b \rho^\nu \sin \nu \left(\pi/3 - \theta\right), \quad \pi/6 < \theta < \pi/3. \tag{15}$$

Substituting Eqs. (14)-(15) into Eqs. (11), (13) one has

$$A_a \cos\frac{\nu\pi}{6} = A_b \sin\frac{\nu\pi}{6} \tag{16}$$

and

$$A_a\sigma_a \sin\frac{\nu\pi}{6} = A_b\sigma_b \cos\frac{\nu\pi}{6}. \tag{17}$$

Relations (16), (17) imply

$$\tan\frac{\nu\pi}{6} = \sqrt{\frac{\sigma_b}{\sigma_a}}. \tag{18}$$

Taking into account that $\sigma_b \ll \sigma_a$ one can easily get

$$\nu \approx \frac{6}{\pi}\sqrt{\frac{\sigma_b}{\sigma_a}}. \tag{19}$$

Now, using the smallness of ν one can obtain from Eq. (8) that

$$A_a \approx 1. \tag{20}$$

Now, using Eq. (7) it is easy to get

$$\sigma = \sqrt{\sigma_a\sigma_b}. \tag{21}$$

Now, we are in a position to apply the obtained formula (21) for a regular isotropic structure with rhombs (see Fig.2). *It is easy to see that if one of the colors corresponds to* the vanishing conductivity, such a structure will not conduct, because this color constitutes "traps". One can represent the total rhombic tesselation as the covering of the plane by identical hexagons, each of which represents a plaquette consisting from alternating grey and black rhombs surrounded by isolating white triangles (see Fig.4).

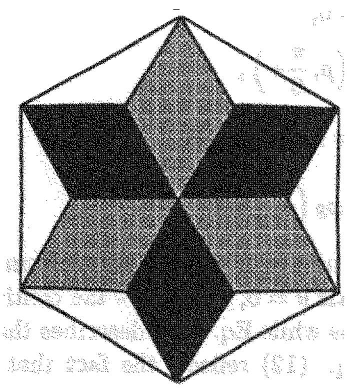

Fig.4. Basic plaquette for the rhombic tesselation

Now, suggesting that white-color regions have some small conductivity $\sigma_1 \ll \sigma_2; \sigma_1 \ll \sigma_3$, we see that the flow of current between plaquettes of the type represented by Fig.4 is realized through two corners of the type represented by Fig.3. Now, using the simple considerations for the network of plaquettes and the formula (21) for the corners, one can get asymptotic formula for the effective conductivity of rhombic tesselation:

$$\sigma_{asymp} = \sqrt{\sigma_1\sigma_2} + \sqrt{\sigma_1\sigma_3}. \tag{22}$$

The similar considerations for the case when $\sigma_1 = \sigma_2 \ll \sigma_3$ gives

$$\sigma_{asymp} = \frac{1}{2}\sqrt{\sigma_1\sigma_3}. \tag{23}$$

The similar result was obtained also by A.M.Dykhne (private communication).

Thus, we have an asymptotic formula for rhombic tesselation in the case when there is the great contrast between conductivities. On the other hand from our preceding work [8], one can understand that for the plane covered by isotropic equal-weighted covering of the plane by N colors, condictivities of which could be written in the form

$$\sigma = 1 + \alpha, \ \langle\alpha\rangle = 0, \tag{24}$$

where α is some small function on the plane (small contrast), the effective conductivity is[1)]

$$\sigma_{eff} = 1 - \frac{1}{N}\langle\alpha^2\rangle + \cdots. \tag{25}$$

Now we can try to construct the tentative formula for the rhomb tesselation which is self-dual, asymptotically true for the case of the large contrast and true for the case of small contrast up to the second order of perturbation theory:

$$\sigma_{rhomb} = \frac{\sqrt{\sigma_1} + \sqrt{\sigma_2} + \sqrt{\sigma_3}}{\frac{1}{\sqrt{\sigma_1}} + \frac{1}{\sqrt{\sigma_2}} + \frac{1}{\sqrt{\sigma_3}}}. \tag{26}$$

Naturally, this formula is only hypothetic one. However, the check of its validity in the next perturbative approximation [8] can give it more solid grounds.

In the paper [9], the authors using self-duality and symmetry of the isotropic three-color equal-weighted tesselations in the plane, have suggested that effective conductivities of such tesselations can be described by the cubic equation which, represents the simple generalization of the well-known Bruggeman effective medium equation [10]:

$$\sigma^3 + A\sigma^2(\sigma_1 + \sigma_2 + \sigma_3) - A\sigma(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3) - \sigma_1\sigma_2\sigma_3 = 0. \tag{27}$$

In the Bruggeman equation $A = 1/3$. The authors have suggested that in this hypothetical equation the constant A is correlated with the geometrical structures. Simple application of this equation to the rhomb tesselation shows that for the case of $\sigma_1 = 0$, the general conductivity is equal to zero and A should be equal to zero. Hence, $\sigma = (\sigma_1\sigma_2\sigma_3)^{1/3}$ for any set of values of $\sigma_1, \sigma_2, \sigma_3$) which obviously contradicts to the asymptotic result (22), obtained above (square roots instead of cubic roots!). Thus, the suggested cubic equation is not universal and is not applicable to the rhomb structure. The source of this misleading lies in the suggestion of the authors [9] that the self-duality and symmetry are enough for the resolution of the three-color problems. However, it is not true and it is well-known from the theory of exactly solvable statistical models[2)]. Even in the frame of cubic equations the idea that A is a constant selects only one from the infinite set of possibilities, because it is easy to see, that A can be an arbitrary function of self-dual combination of three conductivities. In such a way, the cubic equation with the constant A is not a general equation for the symmetric isotropic structures.

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1) Notice that for the multidimensional two-color chessboard we have the same formula, where N is the dimensionality of space.

2) A.B.Zamolodchikov, V.L.Pokrovsky, private communication.

In the conclusion we would like to make one comment regarding other three-color isotropic structure-honeycomb structure consisting of hexagons of three colors (see Fig.11). For such a structure, in the paper [9] the numerical simulation was carried out and the approximate value of the parameter A was found. However, it is obvious that the numerical simulation cannot give the proof of the validity of equation. The plausible check of the formulas for effective conductivity could be the perturbation theory calculations, which should be done at least until sixth order [8].

This work was partially supported by Russian Foundation for Basic Research via grant # 99-02-18409.

Work of A.K. was also supported by CARIPLO Scientific Foundation. We are grateful to J.B.Keller who has attracted our attention to his paper [6]. We thank A.M. Dykhne for useful discussions.

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