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**QUANTUM PARTICLE-TRAJECTORIES AND
 GEOMETRIC PHASE**

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Particle-trajectories are defined as integrable $dx_\mu dp^\mu = 0$ paths in projective space. Quantum states evolving on such trajectories, open or closed, do not delocalise in (x, p) projection, the phase associated with the trajectories being related to the geometric (Berry) phase and the Classical Mechanics action. Properties at high energies of the states evolving on particle-trajectories are discussed.

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Quantal wave-packet revival [1] is the periodic re-assembly of a state's localised structure along a classically stable orbit. The phenomenon has been observed experimentally in Rydberg atoms [2] as well as in one-atom masers [3], and prompts the question whether such revival is possible also for states evolving on open trajectories [4], similarly to classical point-particles. It is shown in this Letter that integrable $dx_\mu dp^\mu = 0$ trajectories in projective space do provide such a context, the aspect being related to the Differential Geometry properties of manifolds [5], independent of the existence of a Hamiltonian.

The revival of quantal wave-packets is connected to the concept of *geometric phase* [6] introduced by Berry. Berry [7] has shown that additionally to a Hamiltonian induced dynamic phase, a quantum state evolving in parameter space on a trajectory that returns to the initial state acquires an extra phase termed geometric phase. Subsequent analysis has generalised the context in which the phenomenon occurs, lifting the restriction of adiabaticity [8], cyclicity and unitarity [9]. An important step was made by the kinematic approach [10], which demonstrated that the Hamiltonian is not needed in defining the geometric phase, and underlined the native geometrical nature of the quantity by relating it to the Bargman invariants [11, 12]. The acquirment of a geometrical phase by quantum states evolving on closed trajectories in parameter space has been verified experimentally in neutron interference [13], in two photon states produced in spontaneous parametric down-conversion [14], etc. The latter paper [14] makes also the important remark that experiments related to non-locality *vis à vis* the Bell inequalities [15] and the Berry phase

are connected, non-locality in Quantum Mechanics being pointed out as a consequence of completeness as early as 1948 by Einstein [16].

The sole assumptions of this Letter are that quantum systems are described by a linear representation space over \mathbb{R} ,¹⁾ and that the coordinate operator x^O has a conjugate operator, $[x^O, k^\nu]_- = -ig^{O\nu} \cdot 1$. The latter operators act as tangent space vectors on the manifold, action revealed by the (Weyl) translation operators $\xi_{\Delta x} \stackrel{\text{def}}{=} e^{+i\Delta x_\mu k^\mu}$ and $\xi_{\Delta k} \stackrel{\text{def}}{=} e^{-i\Delta k_\mu x^\mu}$:

$$\xi_{\Delta x}^\dagger x^O \xi_{\Delta x} = x^O + \Delta c^O, \quad \xi_{\Delta k}^\dagger k^O \xi_{\Delta k} = k^O + \Delta k^O, \quad (1)$$

respectively $|c\rangle \xrightarrow{U_{\Delta x}} |c + \Delta c\rangle$ and $|k\rangle \xrightarrow{U_{\Delta k}} |k + \Delta k\rangle$. Given an arbitrary reference state $|\psi_{ref}\rangle$, a set of translated image-states can be defined as [17]:

$$|\psi(\xi, \kappa)\rangle \stackrel{\text{def}}{=} \xi_{\Delta k} \xi_{\Delta x} |\psi_{ref}\rangle, \quad (2)$$

with correspondingly translated state averages:

$$\begin{aligned} \langle x^O \rangle_{\psi(\xi, \kappa)} &= \langle x^O \rangle_{ref} + \Delta c^O = \xi^O, \\ \langle k^O \rangle_{\psi(\xi, \kappa)} &= \langle k^O \rangle_{ref} + \Delta k^O = \kappa^O. \end{aligned} \quad (3)$$

The spread of the image states is identical to that of the reference state, regardless the $(\Delta c, \Delta k)$ translation:

$$\begin{aligned} \delta c_{\psi(\xi, \kappa)}^O &= \delta c_{ref}^O = \text{const}, \\ \delta k_{\psi(\xi, \kappa)}^O &= \delta k_{ref}^O = \text{const}. \end{aligned} \quad (4)$$

The interchange of $\xi_{\Delta x}$ and $\xi_{\Delta k}$ in the definition of the image state $|\psi(\xi, \kappa)\rangle$ leads to a state corresponding in projective space [8, 12] to the same point, the difference between the two being just a phase factor:

$$\xi_{\Delta x} \xi_{\Delta k} = e^{+i\Delta x_\mu \Delta k^\mu} \xi_{\Delta k} \xi_{\Delta x}. \quad (5)$$

The situation is better evidenced by the comparison of $|\psi_{ref}\rangle$ with its transported image around a $\Delta c \rightarrow \Delta k \rightarrow -\Delta c \rightarrow -\Delta k$ quantum loop:

$$\xi_{loop} = \xi_{\Delta k}^\dagger \xi_{\Delta x}^\dagger \xi_{\Delta k} \xi_{\Delta x} = e^{-i\Delta x_\mu \Delta k^\mu} \cdot 1, \quad (6)$$

respectively around an arbitrary quantum loop:

$$\begin{aligned} \xi_{loop} &= \prod_{loop} \xi_{dk} \xi_{dz} = e^{-i \oint k_\mu dx^\mu} \cdot 1 = \\ &= e^{+i \oint x_\mu dk^\mu} \cdot 1. \end{aligned} \quad (7)$$

In both cases the state acquires a geometrical phase proportional to the (c, k) area enclosed by the loop in projective space. Should this phase be zero, the anholonomy [9]

¹⁾ To be published. In essence it is possible to arrive at the x_μ and k_μ operators and their commutation relation solely on grounds related to separability of states and Differential Geometry properties of manifolds, without prior knowledge of physical equations.

hold preventing the realisation of a proper (x, k) coordinate system on the representation space disappears, as it will be shown in the next paragraph. Generalising equation (5) to continuous open paths:

$$U_{open} = \prod_{initial}^{final} U_{dk} U_{dx} = e^{-i \int_i^f k_\mu dx^\mu} \cdot U_{\Delta k} U_{\Delta x} \quad (8)$$

and holding the initial and final states apart at fixed displacements $(\Delta x, \Delta k)$, a path dependent geometric phase for *open paths* can be defined, arbitrary up to a path independent gauge [18] field $\Phi(x, k)$:

$$S \stackrel{def}{=} - \int_i^f k_\mu dx^\mu = + \int_i^f x_\mu dk^\mu. \quad (9)$$

The above relation supports a class of canonical transformations (such as $Q = k, K = -x$) consistent with $[x^\mu, k^\nu]_- = -ig^{\mu\nu} \cdot \mathbf{1}$ and $\langle x|k \rangle = (2\pi)^{-2} e^{-ix_\mu k^\mu}$, that identifies geometrical phase as the Classical Mechanics *action* [19]. Assuming that $|\psi_{ref}\rangle$ can evolve on two neighbouring paths via a beam-splitter like mechanism, the interference in the final state is destructive unless $\delta S = 0$ (for remote trajectories: $\delta S = 2n\pi$), respectively the *extremal action condition*. Paths satisfying the extremal action condition at each point - or equivalently, in equation (6) $dx_\mu dk^\mu = 0$ - preserve constructive interference along the path, and are termed "*particle-trajectories*" This is not an exclusive category however, non-particle *infodynamics* being equally possible²[20]. The early attempts to formulate Quantum Mechanics in terms of (x, p) coordinates failed due to the non-zero commutator of the coordinate and momentum operators $[x^\mu, p^\nu]_- = -i\hbar g^{\mu\nu} \cdot \mathbf{1}$, and are best summarised by the Heisenberg inequality $\delta x^\mu \cdot \delta p^\nu \geq \frac{1}{2} \hbar g^{\mu\nu}$. Nonetheless, free propagation of quantum systems can be approximated by Classical Mechanics, as hinted by the extremal geometric phase relation above.

Establishing an (x, k) coordinate system on a manifold requires that a translation with a Δx leg followed by one with a Δk leg reach the same point as it would under those operations interchanged:

$$[U_{\Delta x}, U_{\Delta k}]_- = (1 - e^{-i\Delta x_\mu \Delta k^\mu}) U_{\Delta x} U_{\Delta k} = 0. \quad (10)$$

This is possible non-trivially only for spaces at least 2D in dimension, by requiring $\Delta x_\mu \Delta k^\mu = 0$. The problem of establishing an (x, k) grid on a 1D manifold is that a translation around a quantum loop of area $dx \cdot dk = \frac{1}{2}$ accumulates a phase factor π , as seen from equation (7). For manifolds of greater dimension this phase may vanish by *reciprocal phase compensation* among dimensions of opposite metric sign. For an Euclidian metric it can be shown that the condition is met only by trajectories on the sphere, while for the Minkowski metric non-trivial solutions of the $n+1$ pairs of canonically conjugate variables - (Q, K) plus the *temporal* dimension (T, H) - are allowed. To have thus a proper (x, k) coordinate system on the manifold two conditions must be met:

1. **necessary condition:** $dx_\mu dk^\mu = 0 \Big|_{PATH}$.

This relation defines *locally* a coordinate system, and it is better known in physics than apparent at first glance. For example in the case of wave-packet propagation, requiring

² An interesting discussion on the conditions to make semi-classical expansions of path integrals describe quantum systems may be found.

the constituent waves to move in sync yields the condition $\mathbf{v}_g = \nabla_k \omega$, which re-written as $\mathbf{v}_g \cdot d\mathbf{k} = \nabla_k \omega \cdot d\mathbf{k} = d\omega$, becomes:

$$dt \cdot d\omega - dx \cdot d\mathbf{k} = 0. \quad (11)$$

For point-particles, the work-energy relation $dE = \mathbf{F}dx = dx \cdot d\mathbf{p}/dt$ can likewise be re-written as:

$$dt \cdot dE - dx \cdot d\mathbf{p} = 0. \quad (12)$$

2. **sufficient condition:** $d^2x = 0$ and $d^2k = 0$ $\Big|_{PATH}$.

This relation conditions path integrability, necessary for the path independent definition of an (x, k) coordinate system on the manifold. It is a *global* condition, the standard solution³⁾ being – up to a canonical transformation [19]:

$$k_\mu k^\mu = \pm k_C^2, \quad dx^\mu / \|dx\| = k^\mu / k_C. \quad (13)$$

The traditional “dynamical” character of k_μ stems precisely from this solution, and less so from its more distantly related Differential Geometry properties on the manifold. The inertia of the differential equations rules out “cross-over” trajectories from $k_\mu k^\mu > 0$ to $k_\mu k^\mu < 0$ paths, $\pm k_C^2$ being a characteristic of the trajectory. Likewise, trajectories on the light-cone cannot “fall” onto $k_\mu k^\mu > 0$ or $k_\mu k^\mu < 0$ solutions either, due to the gradient of the differential equation parallel to the sheet of the light-cone. The $k_\mu k^\mu = \pm k_C^2$ relation is also well known in physics, in the form of $E = c\sqrt{m_0^2 c^2 + \mathbf{p}^2}$, respectively:

$$(E/c)^2 - \mathbf{p}^2 = (m_0 c)^2. \quad (14)$$

In summary, up to a canonical transformation [19] particle-trajectories provide a ruling of the manifold that satisfies the:

³⁾ For $\|dx\| \neq 0$ the $dx_\mu dk^\mu = 0$ relation requires that dk be “perpendicular” to dx , respectively $dk^\mu = C_\perp^{\mu\nu} \omega_\nu$, where ω is an arbitrary 1-form not “parallel” to the “unit” vector $n^\mu \stackrel{\text{def}}{=} dx^\mu / \|dx\|$ and $C_\perp^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu$ a tensor that selects the “perpendicular” component to dx^μ . To be integrable, dx and dk must be closed forms: $d^2x^\mu = 0$ and $d^2k^\mu = dC_\perp^{\mu\nu} \omega_\nu + C_\perp^{\mu\nu} d\omega_\nu = 0$, where $dC_\perp^{\mu\nu} = -C_\perp^{\mu\rho} dn_\rho n^\nu - n^\mu dn_\rho C_\perp^{\rho\nu}$. The $d^2k = 0$ condition can be re-written as:

$$n^\mu (dn_\rho \wedge dC_\perp^{\rho\nu} \omega_\nu) = C_\perp^{\mu\nu} (d\omega_\nu - n^\rho \omega_\rho \wedge dn_\nu). \quad (A)$$

The left hand side proportional to n and the right hand side “perpendicular” to n imply $C_\perp^{\mu\nu} dn_\mu \wedge \omega_\nu = 0$, condition that has the following solutions: (i) - $C_\perp^{\mu\nu} \omega_\nu = 0$, (ii) - $C_\perp^{\mu\nu} dn_\mu = 0$ and (iii) - $dn_\mu \wedge \omega_\nu = \text{antisymmetric}$. Solution (i) is equivalent to $dk = 0$, solution (ii) restricts dn “parallel” to n – impossible in view of $n_\mu n^\mu = \pm 1$, thus the only viable solution is (iii), $\omega_\nu = k_C \cdot dn_\nu$ where k_C is a scalar field. From the right hand side of equation (A) equal to zero and the arbitrary orientation of dn with respect to n , the scalar field $\Lambda_C = 2\pi/k_C$ must be a constant (known as the “Compton wavelength”). Therefore $dk^\mu = k_C \cdot C_\perp^{\mu\nu} dn_\nu$, or in view of $n_\mu n^\mu = \pm 1$, $dk^\mu = k_C dn^\mu$ and $k^\mu = k_C n^\mu + (\text{const.})^\mu$. In the eigen-system of reference of the trajectory the $dx_\mu dk^\mu = 0$ condition is simply $dk'_0 = 0$. Requiring Lorentz invariance, the constant in the solution above must be zero and:

$$k^\mu = k_C \cdot n^\mu = k_C \cdot \frac{dx^\mu}{d(c\tau)},$$

where τ is the invariant proper-time, $(cd\tau)^2 = \pm dx_\mu dx^\mu$. In the track’s eigen-system of reference $k'_\mu = \pm(k_C, 0)$ and in the laboratory system of reference $k_\mu k^\mu = \pm k_C^2$. The “photonic” case $\|dx\| = 0$ yields $k_\mu k^\mu = 0$. Both in this and in the $\|dx\| \neq 0$ case the solution holds up to a canonical transformation [19].

- translational properties of state averages:

$$\begin{aligned}\langle \mathbf{x}^\mu \rangle_{\psi(\xi, \kappa)} &= \langle \mathbf{x}^\mu \rangle_{ref} + \Delta x^\mu, \\ \langle \mathbf{k}^\mu \rangle_{\psi(\xi, \kappa)} &= \langle \mathbf{k}^\mu \rangle_{ref} + \Delta k^\mu;\end{aligned}\quad (15)$$

- spreadless transport of states:

$$\begin{aligned}\delta x_{\psi(\xi, \kappa)}^\mu &= \delta x_{ref}^\mu = \text{const}, \\ \delta k_{\psi(\xi, \kappa)}^\mu &= \delta k_{ref}^\mu = \text{const};\end{aligned}\quad (16)$$

- x - k evolution⁴⁾ equations:

$$\begin{aligned}d\langle \mathbf{x}^\mu \rangle / \|d\langle \mathbf{x} \rangle\| &= \langle \mathbf{k}^\mu \rangle / k_C, \\ d\langle \mathbf{x}^\mu \rangle d\langle \mathbf{k}_\mu \rangle &= 0, \\ \int_{path} \langle \mathbf{k}_\mu \rangle d\langle \mathbf{x}^\mu \rangle &= \text{extremal};\end{aligned}\quad (17)$$

- path type constraints:

$$\begin{aligned}\langle \mathbf{k}_\mu \rangle \langle \mathbf{k}^\mu \rangle &= \pm k_C^2, \\ \langle \mathbf{k}_\mu \mathbf{k}^\mu \rangle &= \pm k_C^2 - \delta k_\mu \delta k^\mu;\end{aligned}\quad (18)$$

- *contact condition* between the physically meaningful state-averages and the particle-trajectory ruling of the manifold:

$$\begin{aligned}\langle \mathbf{x}^\mu \rangle_{\psi(\xi, \kappa)} &= \xi^\mu, \\ \langle \mathbf{k}^\mu \rangle_{\psi(\xi, \kappa)} &= \kappa^\mu.\end{aligned}\quad (19)$$

Although no physical interpretation has been assumed so far for \mathbf{k} , it is evident that it corresponds to what is more traditionally known as 4-momentum, $\mathbf{p}_\mu = \hbar \mathbf{k}_\mu$.

Since geometric phase properties have been discussed mostly in the context of low energy phenomenae, the following will refer to high energy aspects. Quantum states travelling on particle-trajectories $\langle \mathbf{k}_\mu \rangle \langle \mathbf{k}^\mu \rangle = \text{const.}$ have two constants of motion:

$$\begin{aligned}m_0^2 &\stackrel{\text{def}}{=} \frac{\hbar^2}{c^2} \langle \mathbf{k}_\mu \rangle \langle \mathbf{k}^\mu \rangle, \\ m_{bare}^2 &\stackrel{\text{def}}{=} \frac{\hbar^2}{c^2} \langle \mathbf{k}_\mu \mathbf{k}^\mu \rangle,\end{aligned}\quad (20)$$

the rest and bare mass of the state, related to each other by the spread of the state in k -space:

$$m_{bare}^2 - m_0^2 = \frac{\hbar^2}{c^2} \delta k_\mu \delta k^\mu \quad (21)$$

⁴⁾ Similar to the Ehrenfest theorems, based however on Differential Geometry properties of manifolds, rather than a consequence of evolution equations (i.e. - Schrödinger).

a difference that for most stable systems is negative. The spread of m_{bare}^2 for an evolving quantum state is:

$$\begin{aligned} \langle \delta^2(\mathbf{k}_\mu \mathbf{k}^\mu) \rangle_{\psi(\xi, \kappa)} &= \langle \delta^2(\mathbf{k}_\mu \mathbf{k}^\mu) \rangle_{ref} + \\ &+ 4 \|\Delta k\| \cdot \langle \delta(\mathbf{k}_\mu \mathbf{k}^\mu) \delta(n_\mu \mathbf{k}^\mu) \rangle_{ref} + \\ &+ 4 \|\Delta k\|^2 \cdot \langle \delta^2(n_\mu \mathbf{k}^\mu) \rangle_{ref}, \end{aligned} \quad (22)$$

where $\|\Delta k\| \stackrel{\text{def}}{=} |\Delta k_\mu \Delta k^\mu|^{1/2}$ and $n_\mu = \Delta k_\mu / \|\Delta k\|$. Due to the minimum of the expression in the vicinity of $(\pm m_0 c^2, 0)$ for sub-luminous and $(0, \pm m_0 c)$ for supra-luminous trajectories, the linear term in $\|\Delta k\|$ vanishes and the Klein – Gordon equation holds with good approximation:

$$\mathbf{k}_\mu \mathbf{k}^\mu \simeq \text{const} \cdot 1. \quad (23)$$

For high boost factors $\gamma \rightarrow \infty$ however, the spread in m_{bare}^2 diverges even if $\Delta m_{bare}^{ref} = 0$, the Klein – Gordon equation loosing accuracy:

$$\frac{\Delta m_{bare}}{m_{bare}} \simeq \frac{\hbar \sqrt{2\gamma}}{m_{bare} c} \sqrt{\langle \delta^2(\mathbf{k}_0 - \mathbf{k}_\parallel) \rangle_{min}} \quad (24)$$

as the state approaches the light-cone and overlaps with the densely bunched m_{bare}^2 paths in this region of k -space, as well as with the supra-luminous states across the light-cone. This should be distinguished from seeing the state from a different system of reference (Lorentz boost). The $\Delta m_{bare}/m_{bare}$ magnitude of the effect is on the order of 0.2% for a 1 eV/c wide e^- state accelerated to LEP2 energies, respectively 4% for a 1 MeV/c wide p state accelerated to TEVatron energies. At $E \simeq 300$ GeV a generic 1 eV/c wide e^- state overlaps with hypothetical supra-luminous⁵⁾ components of m_{bare} as high as 0.7 GeV/c².

In summary, $dx_\mu dk^\mu = 0$ integrable trajectories have been shown to transport quantum states non-dispersively in (x, k) projective space. The geometrical phase associated with the trajectories is extremal, its expression being that of the Classical Mechanics action. The trajectories are described by a constant of motion $k_\mu k^\mu = k_C^2$, more traditionally known as the “rest mass”, m_0^2 . Highly boosted quantum states overlap both with higher m_{bare}^2 as well as with negative m_{bare}^2 states.

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⁵⁾ Hypothetical supra-luminous transformations connect transformations across the light-cone, changing the sign of the pseudo-norm. Such type of action interchanges temporal with spatial information parallel to the direction of boost, while rendering arbitrary information perpendicular to it. Supra-luminous transformations would hence obey $\Lambda^\dagger G \Lambda = -C_\parallel G$ and have the form:

$$\Lambda_\nu^\mu = \gamma' \begin{pmatrix} 1 & \beta \\ \beta & C_\parallel \end{pmatrix}$$

where $\gamma' = 1/\sqrt{\beta^2 - 1}$ and C_\parallel a tensor selecting the parallel component to the boost.

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