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CRITICAL EXPONENTS FOR THREE-DIMENSIONAL IMPURE ISING MODEL IN THE FIVE-LOOP APPROXIMATION

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The renormalization-group functions governing the critical behavior of the three-dimensional weakly-disordered Ising model are calculated in the five-loop approximation. The random fixed point location and critical exponents for impure Ising systems are estimated by means of the Padé-Borel-Leroy resummation of the renormalization-group expansions derived. The asymptotic critical exponents are found to be: $\gamma = 1.325 \pm 0.003$, $\eta = 0.025 \pm 0.01$, $\nu = 0.671 \pm 0.005$, $\alpha = -0.0125 \pm 0.008$, $\beta = 0.344 \pm 0.006$, while for the correction-to-scaling exponent less accurate estimate $\omega = 0.32 \pm 0.06$ is obtained.

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Today, two regular field-theoretical methods exist to evaluate the universal critical quantities of uniaxial impure magnets described by the three-dimensional (3D) random Ising model: the $\sqrt{\epsilon}$-expansion invented by Harris and Lubensky [1, 2] and Khmelnitskii [3] and the renormalization-group (RG) approach in three dimensions. The former technique, being well developed [4, 5], was recently shown to have limited numerical power since $\sqrt{\epsilon}$-expansions for critical exponents, calculated starting from the five-loop series [6] up to the $\sqrt{\epsilon}^4$ and $\sqrt{\epsilon}^5$ terms [7], exhibit irregular structure making them unsuitable for subsequent resummation and extracting numerical estimates [8].

On the contrary, the field-theoretical RG approach in three dimensions proved to be very effective when used to estimate the critical exponents and other universal characteristics of the $O(n)$-symmetric systems [9 – 19]. The weakly-disordered Ising model at criticality is known to be described by the n-vector φ^4 field theory with the quartic self-interaction having a hypercubic symmetry, provided $n \to 0$ (the replica limit) and the coupling constants have proper signs. In eighties, the RG expansions for 3D cubic and impure Ising models have been calculated in the two-loop [20], three-loop [21, 22] and four-loop [23, 24] approximations paving the way for estimating the universal critical quantities [20 – 33]. The four-loop 3D RG expansions, however, resummed by the generalized Padé-Borel-Leroy method do not allow to optimize the resummation procedure since there is the only approximant ([3/1]) that does not suffer from positive axis poles. Moreover, an account for the four-loop terms in the 3D RG series shifts the random fixed point coordinates and the correction-to-scaling exponent ω appreciably with respect to the three-loop estimates, indicating that at this step the RG based iterations do not still achieve their asymptote.

In such a situation a calculation of the higher-order contributions to the RG functions looks very desirable. In this Letter, the five-loop RG expansions for the 3D impure Ising model are obtained and the numerical estimates for the critical exponents are found.

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We start from the Landau – Wilson – Hamiltonian of the 3D n-vector cubic model:

$$H = \frac{1}{2}\int d^3x\left[m_0^2\varphi_\alpha^2 + (\nabla\varphi_\alpha)^2 + \frac{u_0}{12}\varphi_\alpha^2\varphi_\beta^2 + \frac{v_0}{12}\varphi_\alpha^4\right], \qquad (1)$$

where φ is an n-component real order parameter, m_o^2 being the reduced deviation from the mean-field transition temperature. In the replica limit, this Hamiltonian describes the impure Ising model provided $u_o < 0$ and $v_o > 0$.

The RG functions for the Hamiltonian (1) are found within a massive theory. To extend known four-loop RG series [23, 24] to the five-loop order, we calculate the tensor (field) factors generated by the cubic interaction. Taking then the values of 3D integrals from Ref.[34], we arrive, under $n = 0$, to the following expansions:

$$\begin{aligned}\frac{\beta_u}{u} = 1 - 8u - 6v + \frac{4(190u^2 + 300uv + 69v^2)}{27} - 199.64042u^3 - 493.84155u^2v - \\ -302.86779uv^2 - 65.937285v^3 + 1832.2067u^4 + 6192.5121u^3v + 6331.2264u^2v^2 + \\ +2777.3942uv^3 + 495.00575v^4 - 20770.177u^5 - 89807.670u^4v - \\ -130340.91u^3v^2 - 90437.636u^2v^3 - 33088.223uv^4 - 5166.3920v^5, \qquad (2)\end{aligned}$$

$$\begin{aligned}\frac{\beta_v}{v} = 1 - 12u - 9v + \frac{4(370u^2 + 624uv + 231v^2)}{27} - 469.33397u^3 - 1228.6059u^2v - \\ -957.78166uv^2 - 255.92974v^3 + 5032.6923u^4 + 17967.851u^3v + 21964.394u^2v^2 + \\ +11856.957uv^3 + 2470.3925v^4 - 64749.282u^5 - 294450.70u^4v - \\ -493917.04u^3v^2 - 407119.31u^2v^3 - 170403.12uv^4 - 29261.585v^5. \qquad (3)\end{aligned}$$

$$\begin{aligned}\gamma^{-1} = 1 - u - \frac{3v}{2} + 2u^2 + 6uv + 3v^2 - 9.4527182u^3 - 42.537232u^2v - 49.298206uv^2 - \\ -16.817754v^3 + 70.794806u^4 + 424.76884u^3v + 752.04939u^2v^2 + 516.26675uv^3 + \\ +130.47743v^4 - 675.69961u^5 - 5067.7471u^4v - 12193.045u^3v^2 - \\ -12966.212u^2v^3 - 6587.8339uv^4 - 1326.2123v^5. \qquad (4)\end{aligned}$$

$$\begin{aligned}\eta = \frac{8(2u^2 + 6uv + 3v^2)}{27} + 0.39494402u^3 + 1.7772481u^2v + 1.9994041uv^2 + \\ +0.66646804v^3 + 6.5121099u^4 + 39.072660u^3v + 68.665263u^2v^2 + 47.140073uv^3 + \\ +11.785018v^4 - 21.647206u^5 - 162.35405u^4v - 382.02381u^3v^2 - \\ -389.99671u^2v^3 - 193.00269uv^4 - 38.600539v^5. \qquad (5)\end{aligned}$$

The five-loop RG series for generic n are presented in Ref.[35].

Numerical values of critical exponents are determined by the coordinates of the random fixed point. To find its location the Padé-Borel-Leroy resummation technique is applied which demonstrated high effectiveness both for $O(n)$-symmetric models [9, 11, 15] and for anisotropic systems preserving their internal symmetries (see, e. g. Ref.[36]). Since the RG functions depend on two variables, the Borel-Leroy transformation is taken in a generalized form:

$$f(u,v) = \sum_{ij} c_{ij} u^i v^j = \int_0^\infty e^{-t} t^b F(ut, vt) dt, \qquad F(x,y) = \sum_{ij} \frac{c_{ij} x^i y^j}{(i+j+b)!}. \tag{6}$$

To perform an analytical continuation, the resolvent series

$$\tilde{F}(x,y,\lambda) = \sum_{n=0}^{\infty} \lambda^n \sum_{l=0}^{n} \frac{c_{l,n-l} x^l y^{n-l}}{n!} \tag{7}$$

is constructed with coefficients being uniform polynomials in u, v and then Padé approximants $[L/M]$ in λ at $\lambda = 1$ are used.

For the resummation of the five-loop RG expansions we employ three different Padé approximants: [4/1], [3/2], and [2/3]. The first of them, being pole free, is known to give good numerical results for basic 3D models of phase transitions, while the others are near-diagonal and should reveal, in general, the best approximating properties. The coordinates of the random fixed point resulting from the series (2, 3) under $b = 0$ and $b = 1$ are presented in Table 1, where superscript "p" stands to mark that the Padé approximant has a "non-dangerous" positive axis pole.

Table 1

	b	[4/1]	[3/2]	[2/3]	[3/1]
U_c	0	−0.7200	−0.7148	−0.6871	−0.6991
	1	−0.7445	-0.7385^p		−0.6839
V_c	0	2.0182	2.0125	2.0571	1.9922
	1	2.0296	2.0236^p		1.9877
ω	0	0.266	0.303	0.462^c	0.376
	1	0.263	0.325^p		0.361

This Table, where widely accepted variables $U = 8u$ and $V = 8v$ are used instead of u and v, contains also the four-loop estimates. The four-loop series were processed on the base of the Padé approximant [3/1], since use of the diagonal approximant [2/2] leads to the integrand in (6) that has a dangerous pole near the random fixed point both for β_u and [2] β_v. The fixed point location given by the approximant [2/3] is presented for $b = 0$ only, because for $b = 1$ this approximation predicts no random fixed point.

As is seen from Table 1, Padé approximants [4/1] and [3/2] yield numerical values of U_c and V_c which are very close to each other. Moreover, for $b = 0$ they are also close to those given by the approximant [3/1]: the largest difference between the five-loop and four-loop estimates does not exceed 0.03. With increasing b corresponding numbers diverge indicating that $b = 0$ is an optimal value of the shift parameter. On the contrary, Padé approximant [2/3] gives the random fixed point location which deviates appreciably from those predicted by approximants [4/1], [3/2], and [3/1]. This approximant, however, leads to poor numerical results even for simpler systems. Indeed, when used to evaluate the coordinate of the Ising fixed point it results in $V_c = 1.475$ (under $b = 0$) while the

[2] In fact, under $b = 1$ the approximant [3/2] generates the expression for β_v that is also spoilt by a positive axis pole at the random fixed point. This pole, however, being well remoted from the origin ($t = 40.12$), turns out to be not dangerous, i. e. does not influence, in practice, upon the evaluation of the Borel integral.

best today estimate is $V_c = 1.411$ [14]. This forces us to reject the data given by the approximant [2/3].

To finally determine the coordinates U_c and V_c, we average the numerical data given by three working Padé approximants at $b = 0$. This procedure yields the values

$$U_c = -0.71, \qquad V_c = 2.01, \tag{8}$$

which are claimed to be the results of our search of the random fixed point location. To estimate their apparent accuracy we accept that deviations of these numbers from the exact ones would not exceed the differences between them and the four-loop results since, among all proper estimates, the four-loop ones most strongly differ from the averaged values. Hence, the error bounds for U_c and V_c are believed to be not greater than ± 0.02. Another way to estimate an apparent accuracy is to trace how the averaged values of the random fixed point coordinates vary with the variation of b. We calculate U_c and V_c using the pole-free approximants [4/1] and [3/1] for b lying between 0 and 15. Running through this interval the averaged coordinates change their values by about 0.02 indicating that an accuracy of the estimates found is of order of few per cents.

Let us evaluate further the critical exponents. The exponent γ is estimated by the Padé-Borel-Leroy summation of the series (4) for γ^{-1} and of the analogous RG expansion for γ, with approximants [4/1] and [3/2] being employed. The numerical value of the Fisher exponent is also found in two different ways: via the estimation of the critical exponent $\eta_2 = (2-\eta)(\gamma^{-1}-1)$ having the RG expansion which exhibits a good summability and by direct substitution of the fixed point coordinates into the series (5) with rapidly diminishing coefficients. Direct summation of the RG expansion for η gives $\eta = 0.027$, numerical results obtained by making use of the resummation procedures just described are collected in Table 2.

Table 2

b		0	1	2	3	5	10	15
$(\gamma^{-1})^{-1}$	[4/1]	1.3236	1.3244	1.3250	1.3254	1.3260	1.3268	1.3272
	[3/2]	–	–	–	1.3253^p	1.3260	1.3265	1.3267
γ	[4/1]	1.3245	1.3248	1.3250	1.3252	1.3254	1.3257	1.3259
	[3/2]	1.3246^p	1.3251^p	1.3254^p	1.3257^p	1.3261^p	1.3267^p	1.3270^p
η	[4/1]	0.0312	0.0276	0.0251	0.0231	0.0204	0.0166	0.0148
(via η_2)	[3/2]	–	–	–	0.0287^p	0.0217^p	0.0167^p	0.0149

In this Table, symbol $(\gamma^{-1})^{-1}$ means that the RG series for γ^{-1} was resummed. The empty cells are due to the dangerous poles spoiling corresponding approximations. The estimates for η standing in the 5-th and 6-th lines were obtained under $\gamma = 1.325$ by the resummation of the RG series for η_2.

As is seen, two methods of evaluating γ lead to remarkably close numerical results which very weakly depend on the tune parameter. Indeed, with increasing b from 0 to 15 the estimates for γ obtained by the resummation of the RG series for γ and γ^{-1} on the base of the pole-free approximant [4/1] vary by less than 0.0036 while the difference between them never exceeds 0.0013. Under the same variation of b, the value of γ averaged over these two most reliable approximations remains within the segment [1.3240, 1.3266]. On the other hand, the accuracy of determination of the critical exponents depends not only on a quality of the resummation procedure but also on the accuracy achieved in the course of locating of the random fixed point. That is why we investigated to what extent the estimates for γ vary when coordinates of the random fixed point run through their error bars. It was found that γ calculated at the optimal value of tune parameter $b = 2$ (see Table 2) does not leave the segment [1.3228, 1.3263]. Hence, the error bounds for the value of γ are believed to be about ± 0.003.

Less stable numerical results are found for the Fisher exponent η. As is seen from Table 2, the values of η given by the RG series for η_2 and the pole-free Padé approximant [4/1] spread from 0.0148 to 0.0312. The average over this interval is equal to 0.023, while the direct summation of the series (5) gives 0.027. Hence, 0.025 should play a role of the most likely value of exponent η. Since the estimates for η found via the evaluation of η_2 are sensitive to the accepted value of γ the apparent accuracy achieved in this case is not believed to be better than ± 0.01.

Having estimated γ and η, we evaluate other critical exponents using well-known scaling relations. The final results of our five-loop RG analysis are as follows:

$$\gamma = 1.325 \pm 0.003, \qquad \eta = 0.025 \pm 0.01, \qquad \nu = 0.671 \pm 0.005,$$
$$\alpha = -0.0125 \pm 0.008, \qquad \beta = 0.344 \pm 0.006. \tag{9}$$

It is interesting to compare these numbers with those obtained earlier within the lower-order RG approximations. For the exponent γ previous 3D RG calculations gave the values 1.337 [20, 25] (two-loop), 1.328 [22] (three-loop), 1.326 [23] (four-loop), and 1.321 [24] (four-loop). Being found by means of the different resummation procedures, they are, nevertheless, centered around our estimate which is thus argued to be close to the exact value of γ or, more precisely, to the true asymptote of the RG iterations.

In conclusion, we evaluate the correction-to-scaling exponent ω. This exponent is known to be equal to the eigenvalue of the stability matrix

$$\begin{pmatrix} \dfrac{\partial \beta_u}{\partial u} & \dfrac{\partial \beta_u}{\partial v} \\ \dfrac{\partial \beta_v}{\partial u} & \dfrac{\partial \beta_v}{\partial v} \end{pmatrix} \tag{10}$$

that has a minimal modulus. The derivatives entering this matrix are evaluated numerically at the random fixed point on the base of the resummed RG expansions for β_u and β_v and then the matrix eigenvalues are found. Such a procedure leads to the estimates for ω presented in Table 1 (lower lines); the superscript "c" denotes that ω is complex within corresponding approximation and its real part is presented. The numerical values obtained are seen to be considerably scattered and sensitive to the tune parameter. The average over three working Padé approximants, however, being equal to 0.315 at $b = 0$ and to 0.316 at $b = 1$ turns out to be stable under the variation of b unless b becomes large. It is natural therefore to accept that

$$\omega = 0.32 \pm 0.06. \tag{11}$$

This number is smaller by $0.05 - 0.07$ than its counterparts given by recent Monte-Carlo simulations [37] and the alternative RG analysis [32], but their central values lie within the declared error bounds (11). Hence, an agreement between the results discussed exists. On the other hand, the estimate just found needs to be refined, along with the estimates for η and α also exhibiting appreciable uncertainties. Hopefully, a proper processing of the six-loop expansions obtained very recently [38] would enable one to improve further the accuracy of the predictions given by the field-theoretical RG approach.

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