Pis'ma v ZhETF, vol.70, iss.3, pp.167 - 170

IMPROVEMENT OF PERTURBATION THEORY IN QCD FOR e^+e^- →HADRONS AND THE PROBLEM OF α_s FREEZING

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Submitted 15 June 1999

We develope the method of improvement of perturbative theory in QCD, applied to any polarization operator. The case of polarization operator $\Pi(q^2)$, corresponding to the process e^+e^- →hadrons is considered in details. Using the analytical properties of $\Pi(q^2)$ and perturbative expansion of $\Pi(q^2)$ at $q^2 < 0$, $\mathrm{Im}\Pi(q^2)$ at $q^2 > 0$ is determined in such a way, that the infared pole is eliminated. The convergence of perturbative series for $R(q^2) = \sigma(e^+e^- \to \text{hadrons})/(e^+e^- \to \mu^+\mu^-)$ is improved. After substitution of $R(q^2)$ into dispersion relation the improved Adler function $D(q^2)$ with no infrared pole and frozen $\alpha_s(q^2)$ has been obtained. A good agreement with experiment has been achieved.

PACS: 12.38.Cy

It is well known for years, that perturbative calculations of amplitudes in field theories are legitimate, if virtualities of all external legs k_i^2 are negative, $k_i^2 < 0$. In this case polarization operators and vertex functions (two- and three-point functions) are off mass shell and have no singularities. However, in order to get physical predictions, it is necessary to go to $k_i^2 > 0$ (at least for some of them). This can be achieved by analytical continuation using known analytical properties of amplitudes. A typical example is e^+e^--annihilation into hadrons in QCD. The total cross section $\sigma(e^+e^- \to \text{hadrons})$ is proportional to imaginary part of the polarization operator $\Pi_{\mu\nu}(q^2)$ of virtual photon at $q^2 > 0$. At large enough $q^2 < 0$ $\Pi_{\mu\nu}(q^2)$ can be calculated perturbatively in QCD in terms of the expansion over the running coupling constant $\alpha_s(q^2) = 4\pi/\beta_0 \ln(-q^2/\Lambda^2)$. The tensor structure of $\Pi_{\mu\nu}(q)$ is

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2\delta_{\mu\nu})\Pi(q^2) \tag{1}$$

and $\Pi(q^2)$ is an analytical function of q^2 in the whole q^2 complex plane with a cut along positive q^2 semiaxes. Analytical continuation from $q^2 < 0$ to $q^2 > 0$ results in substitution of $\ln(q^2/\Lambda^2) \to \ln(Q^2/\Lambda^2) - i\pi$, $Q^2 = \mid q^2 \mid$. Since small α_s corresponds to large $\ln(Q^2/\Lambda^2)$, the standard procedure (see, e.g. [1]) is to consider $\ln(Q^2/\Lambda^2)$ as large comparing with π and to perform the expansion in $\delta = \pi/\ln(Q^2/\Lambda^2)$. However, practically it is not a good expansion parameter. At typical scale of e^+e^--annihilation, $Q^2 \sim 10$ GeV^2 and $\Lambda \sim 300 - 400$ MeV, $\ln(Q^2/\Lambda^2) \approx 4 - 5$ and $\delta \approx 0.7$.

In this paper we present the systematic method of improvement of perturbation theory in QCD, which is free from this drawback. Besides e^+e^--annihilation, this method can be applied to any polarization operators, for example, to those used in the QCD sum rule approach. The idea of the method has been suggested by Radyushkin [2] and considered also by Pivovarov [3], but many important features and consequences of this method

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were not touched in [2],[3] (particularely, eq. (10) below was not obtained and analysed). Consider the Adler function

$$D(q^2) = -q^2 \frac{d\Pi(q^2)}{dq^2} = -q^2 \int\limits_0^\infty \frac{R(s)ds}{(s-q^2)^2}, \quad (2)$$

where $R(s) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$. In the parton model $R(s) = R_p = 3\sum\limits_q e_q^2$, where e_q is the charge of the quark of flavour q. It is convenient to write:

$$D(q^2) = R_p(1 + d(q^2)), \quad R(q^2) = R_p(1 + r(q^2)), \quad \Pi(q^2) = R_p(1 + p(q^2)). \quad (3)$$

From (3), there follows the equation:

$$d(q^2) = -q^2 \frac{dp(q^2)}{dq^2}. \quad (4)$$

The solution of (4) is

$$p(q^2) - p(\mu^2) = -\int\limits_{\mu^2}^{q^2} \frac{ds}{s} d(s), \quad (5)$$

$r(q^2)$ is proportional to discontinuity of $p(q^2)$ at $q^2 > 0$:

$$r(q^2) = \frac{1}{\pi} \mathrm{Im} p(q^2) = \frac{1}{2\pi i} \left[p(q^2 + i\varepsilon) - p(q^2 - i\varepsilon) \right]. \quad (6)$$

At negative $q^2 < 0$, $q^2 = -Q^2$ the perturbative expansion of $d(Q^2)$ in $\overline{MS}$ renormalization scheme is known up to the third order [4],[5]:

$$d(Q^2) = a(1 + d_1 a + d_2 a^2), \quad d_1 = 1.986 - 0.115 N_f,$$

$$d_2 = 18.244 - 4.216 N_f + 0.086 N_f^2, \quad (7)$$

where $a(Q^2) = \alpha_s(Q^2)/\pi$, N_f is the number of flavours and the small gluon-gluon scattering terms are omitted. With the same accuracy, three loops expression for $\alpha_s(Q^2)$ in $\overline{MS}$ is given by [6, 7]

$$a(Q^2) = \frac{4}{\beta_0 L} \left\{ 1 - \frac{2\beta_1}{\beta_0^2} \frac{\ln L}{L} + \frac{4\beta_1^2}{\beta_0^4 L^2} \left[(\ln L - \frac{1}{2})^2 + \frac{\beta_2 \beta_0}{8\beta_1^2} - \frac{5}{4} \right] \right\}, \quad (8)$$

where $L = \ln(Q^2/\Lambda^2)$ and

$$\beta_0 = 11 - \frac{2}{3} N_f, \quad \beta_1 = 51 - \frac{19}{3} N_f, \quad \beta_2 = 2857 - \frac{5033}{9} N_f + \frac{325}{27} N_f^2. \quad (9)$$

Substitution of (7), (8) into (5) and (6) leads to perturbative corrections up to the third order in the physically measurable quantity $r(q^2)$.By taking the discontinuity, any dependence on the normalization point μ^2 is eliminated. It should be stressed, that in such calculation no expansion in $\pi/\ln(Q^2/\Lambda^2)$ is performed, the only assumptions used(in fact, they are not assumptions, but theorems), are the analytical properties of $\Pi(q^2)$. The result is ($q^2 \geq 0$):

$$r(q^2) = \frac{4}{\pi\beta_0}\,\kappa - 8\frac{\beta_1}{\beta_0^3}\frac{1}{\pi^2\tau^2}\left[\ln(\pi\tau) + 1 - \kappa t\right] + \left(\frac{4}{\beta_0}\right)^2 \frac{d_1}{\pi^2}\,\frac{1}{1+t^2} +$$

$$+16\frac{\beta_1^2}{\beta_0^5}\,\frac{1}{\pi^3\tau^4}\left\{\left(\frac{\beta_2\beta_0}{8\beta_1^2} - 1\right)t + \kappa(1-t^2)\ln(\pi\tau) + t[\ln^2(\pi\tau) - \kappa^2]\right\} -$$

$$-4d_1\left(\frac{4}{\beta_0^2}\right)^2 \frac{\beta_1}{\pi^3\tau^4}\left\{t\left[\ln(\pi\tau) + \frac{1}{2}\right] + \frac{1}{2}\kappa(1-t^2)\right\} + d_2\left(\frac{4}{\beta_0}\right)^3 \frac{t}{\pi^3\tau^4}, \qquad (10)$$

where

$$t = \frac{1}{\pi}\ln\frac{q^2}{\Lambda^2}, \quad \tau = (1+t^2)^{1/2}, \quad \kappa = \frac{\pi}{2} - \mathrm{arctg}\, t. \qquad (11)$$

Essential features of (10) are: 1) unlike standard perturbation expansion $r(q^2)$ has no infra-red poles at $q^2 = \Lambda^2$ and tends to finite limit at $q^2 \to 0$, $r(0) = 4/\beta_0 \approx 0.414$, corresponding to $\alpha_s(0) = 4\pi/\beta_0 \approx 1.3$, all higher order terms vanish; 2) the convergence of perturbation series in (10) is much better, than of the standard ones: the ratio of the second order term to the first order one is smaller than 0.2 everywhere and the ratio of the third order term to the second one is smaller than 0.1, while in the standard approach the ratio of the third order term to the second order one exceeds 0.5 below $q^2 = 10$ GeV^2 and is larger than 1 at $q^2 \to 1$ GeV^2; 3) the α_s corrections, given by (10), are remarkably smaller in the low energy domain, $q^2 \to 5$ GeV^2, than the standard ones, e.g., at $q^2 = 1$ GeV^2 $r(\text{eq.10})/r_{stand} = 0.72$.

In order to get $r(q^2)$ at some lower values of q^2, starting from the value of α_s at Z-boson mass, $\alpha_s(m_z)$, which is rather well known now, it is necessary to use renormalization group equations (8) and perform matching at the thresholds of new flavours production b and c quarks and, if we would like to go to $q^2 < 1$ GeV^2, even at s-quark threshold. This matching procedure introduces some uncertainty. The matching may be performed at $2m_q$ (or, what is practically equivalent, at m_Υ and $m_{J/\psi}$) or at m_q. The former seems to be preferable in the case of e^+e^--annihilation for the evident reasons. There are some arguments in the favour of the latter choice, based on minimal sensitivity of the results to the matching point value [8]. In the standard approach the results are rather sensitive to the choice of matching points – the ratio of r's in the two above mentioned cases is about 1.10 – 1.15 at $q^2 \to 5$ GeV^2. We calculated the q^2 dependence of $r(q^2)$ starting from the point $q^2 = m_z^2$ and going down. The value $r(m_z^2)$ was found from the requirement, that $\alpha_s(m_z^2)$ determined in the standard way is equal $\alpha_s(m_z^2) = 0.119 \pm 0.002$ [9]. Then $\Lambda_5 = 230^{+27}_{-27}$ MeV was found. In the evolution down to lower energies the matching of $r(q^2)$ at the masses of Υ, J/ψ and φ was performed resulting in $\Lambda_4 = 335^{+35}_{-33}$ MeV, $\Lambda_3 = 414^{+41}_{-37}$ MeV, $\Lambda_2 = 490^{+51}_{-47}$ MeV. It was found particularly, that $\pi r(1\ \text{GeV}^2) = 0.41$. If, instead of matching at quarkonium masses the matching at m_q would be done, the values of $r(q^2)$ would be only 4% smaller below $q^2 = 10$ GeV^2 practically independent of Q^2. Therefore, in this aspect this method has also some advantages. The performed comparison with experiment demonstrated a good agreement, starting from $\sqrt{q^2} = 0.7$ GeV, i.e. in much broader interval, than in recent paper [10].

After substituting $r(q^2)$, given by (10) into dispersion relation (2) we get back Adler function $D(q^2)$. Obtained in this way improved $D(q^2)_{impr}$ has correct analytical properties and no unphysical singularities. Therefore, the adopted here procedure has a serious advantage: the required analytical properties of Adler function are restored. If $D(q^2)_{impr}$ is represented in terms of improved effective QCD coupling constant $\alpha_s(q^2)_{impr}$, this would mean, that $\alpha_s(q^2)_{impr}$ has no infrared pole and is frozen at $q^2 \to 0$. In this aspect our approach has some resemblance to the approach by Shirkov and Solovtsov [11] (see also [12]), where the condition of analyticity of $\alpha_s(q^2)$ in the cutted q^2 complex plane was imposed. The difference is, that we exploit the analyticity of polarization operator $\Pi(q^2)$, which is a strict result in field theory and do not use any hypothesis about analyticity of $\alpha_s(q^2)$.

The method to define the improved QCD coupling constant $\alpha_s(q^2)_{impr}$ through $D(q^2)_{impr}$ looks very promisable. For this goal, perhaps, the most suitable is to use not $\overline{MS}$, but the Brodsky, Lepage, Mackenzie renormalization scheme [13], where e^+e^--annihilation is considered as a basing process for definition of $\alpha_s(q^2)$ with no higher order α_s corrections. This problem requires further investigation.

The presented above method can be applied to the treatment of perturbative corrections to any polarization operators and, therefore may result to improvement of QCD sum rule approach.

This work was supported in part REBR grant 97-02-16131.

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